Infinitesimal Fundamentum

Buğra Kabil

Abstract. We present a new approach for the concept of the derivative initiated by Sir Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century. We extend the ideas of Newton and Leibniz about the derivative by introducing an infinitesimal fundamentum. The infinitesimal fundamentum of an one-dimensional real-valued function is a mapping that coincides with the function in two points where the difference of these points is infinitely small. In particular, for a differentiable function its derivative is conform to the derivative of the infinitesimal fundamentum. This yields several equivalent definitions for the derivative of real-valued functions. The consequences are new expansion formulae obtained by generic (in particular nonlinear) infinitesimal fundamenta. That means, we present new expansion formulae for arbitrary functions which are differentiable up to an order. This new approach of infinitesimal fundamentum generalizes the understanding of known expansion formulae as Taylor's formula which is obtained by a linear infinitesimal fundamentum. Especially, we determine a new formula through an exponential infinitesimal fundamentum.

There are many applications in applied analysis and mathematical physics. We present a comparison of several approximation formulae resulting from different infinitesimal fundamenta. The choice of an optimal approximation depends on the infinitesimal fundamentum and on the considered function itself. Next, we suggest a new algorithm for finding the zeros of real-valued functions. This method can be seen as a modification of Newton's method. We show examples where this new method is more effective than Newton's. We also show that the new kind of definitions for the derivative allows to construct a new and non-classical numerical scheme for differential equations. Relying on the finite difference method one can now approximate the derivative in several ways to obtain discretized equations. Depending on the differential equations we can choose a suitable infinitesimal fundamentum to get a entropy consistent discretization. Further, in case of convergence the expansion formulae yield for smooth functions at chosen points convergent series of real numbers. Some series are presented and the values are determined through the new found expansion formulae.

This work gives a new insight into the comprehension for the derivative of one-dimensional functions. Certainly, there are resulting unsolved problems. We state a list of open questions and conjectures.

Keywords · Infinitesimal Calculus · Derivative · Nonstandard Analysis · Infinitesimal Fundamentum · Hyperreal Numbers · Taylor's Theorem · Expansion Formula · Mean Value Theorem · Approximation · Newton's Method · Numerical Scheme · Entropy Consistent Approximations of Conservation Laws in Time and Space · Series ·

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1 Introduction

Infinitesimal calculus has been initiated by Sir Isaac Newton and Gottfried Wilhelm Leibniz in the 17th century, see the pioneering works [23,19]. Calculus with infinitely small and infinite numbers has been used by many mathematicians in the 17th and 18th century like Leonhard Euler in [11]. This approach developed the concept of the derivative of a function. Newton and Leibniz introduced in two different ways the precursor definition of the derivative. In the 19th century the calculus with infinitesimals was replaced by the *limit* and *epsilon-delta* representation according to [4,28]. This results into the well-known definition of the derivative for a real-valued function $f : \mathbb{R} \to \mathbb{R}$ in some point $x_0 \in \mathbb{R}$ which reads as the existence of the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \,. \tag{1}$$

We consider the original ideas of Newton and Leibniz about the derivative and extend the definition in an equivalent way. To do so, we use the theory of Nonstandard Analysis following [18,14]. In the 20th century the theory of infinitesimal calculus was revived as Nonstandard Analysis and introduced in a rigorous way by Abraham Robinson in [26]. This allows to introduce the infinitesimal fundamentum (see Definition 6) of a real-valued function. The infinitesimal fundamentum is a functions which enables the definition of the derivative. For example, a special exponential infinitesimal fundamentum (see Definition 8) yields the limit

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{e^h-1} \tag{2}$$

which is the derivative of a differentiable function f in $x_0 \in \mathbb{R}$ by the rule of l'Hospital. Obviously, the quotient in definition in (2) goes faster to the derivative as in (1) for rapidly increasing and differentiable functions. In particular, a special linear infinitesimal fundamentum (see Definition 7) implies the standard definition (1).

The quotient in the limit (1) can be seen as an approximation of the derivative for $|h| \ll 1$. Rearranging yields an approximation formula for f at the point $x_0 + h$

$$f(x_0+h) \approx f(x_0) + f'(x_0)h$$

or according to (2)

$$f(x_0+h) \approx f(x_0) + f'(x_0)(e^h-1)$$

for $x_0 \in \mathbb{R}$ and $|h| \ll 1$. Taylor's Theorem expand this approximation to the formula

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2!}h^2 + \frac{f^{(3)}(x_0)}{3!}h^3 + \cdots$$

where higher order derivatives of the considered function f are included, see for example [17, 13, 2, 1]. We refine Taylor's formula concerning to an arbitrary infinitesimal fundamentum. For instance, in relation to (2), we obtain an expansion formula of the form

$$f(x_0 + h) = c_0(x_0) + c_1(x_0)(e^h - 1) + c_2(x_0)(e^h - 1)^2 + c_3(x_0)(e^h - 1)^3 + \cdots$$

for some to be determined quantities $c_j(x_0)$ (for $j \in \mathbb{N}_0$) depending on the derivatives of the considered smooth function. The explicit determination of $c_j(x_0)$ is stated as the main result of Section 3 in Theorem 1. Note that $c_j(x_0)$ does not coincide for all $j \in \mathbb{N}_0$ with the terms from Taylor's formula. The new derived formula in Theorem 1 yields a triangular array of numbers resulting from an exponential infinitesimal fundamentum, see Figure 4 in Section 3 for the explicit representation.

We conclude from the new expansion formulae (obtained by different infinitesimal fundamenta) several applications, see Section 4. In mathematics and mathematical physics it is an advantage to approximate given functions to obtain quantitative results, see e.g. [15, 16, 8, 6]. We mention the description of a mathematical pendulum which reads as a nonlinear ordinary differential equation including the sine. For small angular displacements the mathematical pendulum is approximated by a linear equation where the approximation is done via Taylor's formula, see [15, 16]. Depending on the considered function one can now choose several approximations around a given point resulting from different infinitesimal fundamenta. The error difference of the function and its approximation varies from the chosen approximation formula. An effective choice of the formula depends on the properties of the considered function or respectively of the solution to differential equations. The approximation formula can also be helpful for finding the roots of real-valued functions. We mention the explicit representation of the square root of two which is a transcendent number. The property how fast the quotient in (2) goes to the derivative becomes important if we consider an algorithm for finding the roots of real-valued functions. In Section 4 we present a faster iterative algorithm than Newton's method for finding the square root of two.

The mathematical description of physical processes at the continuum-mechanical level often leads to systems of partial differential equations, see for instance [12, 25, 7]. On the microscopic level the description is given by a typically high dimensional system of ordinary differential equations. It is also useful to study an approximation of differential equations as for the description of the mathematical pendulum. Another way to approximate differential equations is to consider the system of equations on a grid. This yields a numerical discretization which converges in the limit case to the original system. In particular, the theory of numerical analysis uses the approximation of the derivative to obtain discretized systems. Now, we have the possibility to choose several different discretizations of differential equations. Note that the convergence of the expansion formulae and so the efficiency of the numerical schemes are not discussed in this work. In the case that the expansion of smooth functions implies a convergent series of real-valued functions, we obtain for fixed points convergent series of real numbers.

1.1 Outline

This work is organized as follows. In Section 2 we briefly present the theory of Nonstandard Analysis. We define an equivalence relation on the space of real-valued sequences $\mathbb{R}^{\mathbb{N}}$. The extension of real numbers to the hyperreal numbers is done by a quotient set with the use of the previous equivalence relation. This deduces the definition of infinitely small and infinite numbers. Then, the ideas of Newton and Leibniz about the derivative are presented. The concept of infinitesimals allows to extend the ideas of Newton and Leibniz to define the infinitesimal fundamentum of a real-valued function. This section is closed with several examples for the infinitesimal fundamentum and hence the resulting definition of the derivative. Section 3 contains the new expansion formula resulting from an exponential infinitesimal fundamentum. The main result in this section is formulated in Theorem 1. The proof of this theorem is technical and will be done by an inductive way. The expansion formula resulting from a slow exponential infinitesimal fundamentum is stated in Theorem 2. The proof of Theorem 2 is moved to Appendix A. This section is closed with a comparison of the given expansion formulae for one example. In Section 4 we present several applications of the new derived expansion formulae. We start with an approximation of functions around the expansion point. We state several approximations resulting from different infinitesimal fundamenta. Then, we present a method for finding the roots of function and compare this algorithm with Newton's method for one example. After that non-classical numerical schemes for partial differential equations are introduced. This section is closed with the application of the expansion formulae to series. In the last section we predict some conjectures and state open questions.

2 Infinitesimal Calculus

The goal of this section is to introduce the definition of the infinitesimal fundamentum of a real-valued function. For this definition we need the theory about the calculus with infinitesimal numbers. Many mathematicians like Leonhard Euler used this kind of numbers in the 18th century. As a motivation we state the original derivation of Euler's number from Euler's work [11, Chapter 7] by using infinitesimal numbers.

> It is known that $a^0 = 1$. If a is a number greater than one then it follows for varying the exponent of a with an infinitely small number ω that there is an infinitely small number ψ such that
>
> $$a^\omega = 1 + \psi\,.$$
>
> Since a is unknown we set $\psi = k\omega$ for some real number k. It yields for an arbitrary number i (Note that this is not the imaginary number) that
>
> $$a^{i\omega} = (1+k\omega)^i = 1 + \frac{i}{1}k\omega + \frac{i(i-1)}{2!}k^2\omega^2 + \frac{i(i-1)(i-2)}{3!}k^3\omega^3 + \cdots\,,$$
>
> where this expansion was well known. We choose $i = \frac{z}{\omega}$ for some finite number z, i.e. i is now chosen as an infinitely large number and obtain
>
> $$a^z = 1 + \frac{1}{1}kz + \frac{(i-1)}{1\cdot 2i}k^2z^2 + \frac{(i-1)(i-2)}{1\cdot 2i\cdot 3i}k^3z^3 + \cdots\,.$$
>
> Since i is an infinitely large number, it follows that $\frac{i-1}{i} = 1$, $\frac{(i-1)}{2i} = 1/2$ and so on. We obtain for $z = 1$
>
> $$a = 1 + \frac{1}{1}k + \frac{1}{2!}k^2 + \frac{1}{3!}k^3 + \frac{1}{4!}k^4 + \cdots$$
>
> and especially we define for $k = 1$ the finite number
>
> $$a = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\,.$$
>
> *abstracted from* [11]: INTRODUCTIO IN ANALYSIN INFINITORUM, *Chapter 7*, 1748

In the 19th century the calculus with infinitesimals was replaced by limits, respectively by the epsilon-delta method. In the 20th century the theory about Nonstandard Analysis was founded where real numbers are extended to hyperreal numbers which allows the calculus with infinitesimals in a rigorous way.

Next, we give an introduction into the theory of Nonstandard Analysis to understand the unknown properties of functions varying in infinitely small quantities.

2.1 Nonstandard Analysis

We briefly introduce the essential basics of Nonstandard Analysis which will be used in this work, see [14, 18] for the details. The formal definition of the hyperreal numbers $\mathbb{R}^*$ is given by the quotient set

$$\mathbb{R}^* := \mathbb{R}^{\mathbb{N}} / \sim$$

where the equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$ will be defined with the help of an ultrafilter on $\mathbb{N}$. We are going to introduce the definition of an ultrafilter and the equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Definition 1. A nonempty system $\mathcal{F} \subset \mathcal{P}(\mathbb{N}) = \left\{A \,\middle|\, A \subset \mathbb{N}\right\}$ is called a ***filter*** on $\mathbb{N}$ if the following holds

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}$, $A \subset B \subset \mathbb{N} \Rightarrow B \in \mathcal{F}$.

An ***ultrafilter*** is a filter that satisfies

$$\forall\, A \subset \mathbb{N} \Rightarrow (A \in \mathcal{F} \;\vee\; A^c \in \mathcal{F}) \,,$$

where $A^c := \mathbb{N} \setminus A$. The ***cofinite filter*** on $\mathbb{N}$ is the system

$$\mathcal{F}_0 := \left\{A \subset \mathbb{N} \,\middle|\, \mathbb{N} \setminus A \text{ is finite}\right\} .$$

Zorn's Lemma guarantees the existence of an ultrafilter including the cofinite filter. For our analysis we fix such an ultrafilter on $\mathbb{N}$ with

$$\mathbb{N} \setminus A \in \mathcal{F} \text{ for all finite subsets } A \subset \mathbb{N}.$$

Then, we can define an equivalence relation $\sim$ on $\mathbb{R}^{\mathbb{N}}$.

Definition 2. For $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and $\beta = (\beta_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ we define the ***equivalence relation*** on $\mathbb{R}^{\mathbb{N}}$ through

$$\alpha \sim \beta \;:\Leftrightarrow\; \left\{n \in \mathbb{N} \,\middle|\, \alpha_n = \beta_n\right\} \in \mathcal{F} .$$

For a real number $r \in \mathbb{R}$ we define the ***constant sequence*** $r_{\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ through

$$r_{\mathbb{N}} := (r, r, r, \ldots) .$$

The ***equivalence class*** $[\alpha]$ for any $\alpha \in \mathbb{R}^{\mathbb{N}}$ is

$$[\alpha] := \begin{cases} r & \text{if } \alpha \sim r_{\mathbb{N}} \text{ for some } r \in \mathbb{R}, \\ \left\{\beta \in \mathbb{R}^{\mathbb{N}} \,\middle|\, \alpha \sim \beta\right\} & \text{otherwise} . \end{cases}$$

The ***hyperreal numbers*** $\mathbb{R}^*$ are defined by

$$\mathbb{R}^* := \left\{[\alpha] \,\middle|\, \alpha \in \mathbb{R}^{\mathbb{N}}\right\}$$

Remark 1. Note that the construction of the hyperreal numbers $\mathbb{R}^*$ depends on the given ultrafilter and we have the following properties

- $\mathbb{R} \subset \mathbb{R}^*$,
- $\forall r \in \mathbb{R} : [r_{\mathbb{N}}] = r$,
- $[\alpha] = [\beta] \;\Leftrightarrow\; \alpha \sim \beta$ for $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$.

Proposition 1. *The structure $(\mathbb{R}^*, +, \cdot, \leq)$ is an ordered field with zero $[0_{\mathbb{N}}] = 0$ and unity $[1_{\mathbb{N}}] = 1$, where we have the following definitions:*

(i) for $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$ we define

$$[\alpha] + [\beta] := [\alpha + \beta] \qquad \textit{and} \qquad [\alpha] \cdot [\beta] := [\alpha \cdot \beta] \,,$$

(ii) for $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$ we define

$$[\alpha] \leq [\beta] \;:\Leftrightarrow\; \left\{n \in \mathbb{N} \,\middle|\, \alpha_n \leq \beta_n\right\} \in \mathcal{F} .$$

Now, the goal is to define infinitesimals and an equivalence relation $\simeq$ on the hyperreal numbers $\mathbb{R}^*$. We are going to introduce some useful notations.

Definition 3. Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$. We say

$$\alpha_j = \beta_j \text{ almost everywhere (a.e.) } :\Leftrightarrow \left\{ j \in \mathbb{N} \,\middle|\, \alpha_j = \beta_j \right\} \in \mathcal{F}$$

and

$$\alpha_j \leq \beta_j \text{ a.e. } :\Leftrightarrow \left\{ j \in \mathbb{N} \,\middle|\, \alpha_j \leq \beta_j \right\} \in \mathcal{F}\,.$$

In an analogous way we define $\alpha_j \geq \beta_j$ almost everywhere. Further, we define for $\alpha \in \mathbb{R}^{\mathbb{N}}$ the ***absolute value*** through

$$|[\alpha]| := \begin{cases} [\alpha] & \text{for} \quad [\alpha] \geq 0\,, \\ -[\alpha] & \text{for} \quad [\alpha] \leq 0\,. \end{cases}$$

We state some properties for the calculus in $\mathbb{R}^*$ where the proofs can be found in [14, 18].

Proposition 2. *Let $\alpha, \beta \in \mathbb{R}^{\mathbb{N}}$ and $r, \epsilon \in \mathbb{R}$ with $\epsilon > 0$. Then, the following holds*

(i) $[\alpha] = [\beta] \quad \Leftrightarrow \quad \alpha_j = \beta_j$ *a.e.,*

(ii) $|[\alpha] - [\beta]| \leq \epsilon \quad \Leftrightarrow \quad |\alpha_j - \beta_j| \leq \epsilon$ *a.e.,*

(iii) $\lim\limits_{j\to\infty} \alpha_j = r \in \mathbb{R} \quad \Rightarrow \quad |[\alpha] - r| \leq 1/n$ *for all* $n \in \mathbb{N}$*,*

(iv) $\lim\limits_{j\to\infty} \alpha_j = +\infty \in \mathbb{R} \quad \Rightarrow \quad [\alpha] \geq n$ *for all* $n \in \mathbb{N}$*,*

Definition 4. Let $[\alpha], [\beta] \in \mathbb{R}^*$.

(i) $[\alpha]$ is called ***finite*** if $[\alpha] \leq n$ for some $n \in \mathbb{N}$,
(ii) $[\alpha]$ is called ***infinite*** if $[\alpha] \geq n$ for all $n \in \mathbb{N}$,
(iii) $[\alpha]$ is an ***infinitesimal*** if $[\alpha] \leq 1/n$ for all $n \in \mathbb{N}$,
(iv) We define an ***equivalence relation*** on $\mathbb{R}^*$ through

$$[\alpha] \simeq [\beta] \ :\Leftrightarrow \ [\alpha] - [\beta] \text{ is an infinitesimal.}$$

Note that $\mathbb{R}^*$ includes infinitesimals which are not equal to zero, e.g. $[(\frac{1}{j})_{j\in\mathbb{N}}]$ is an infinitesimal and not equal zero. We also have for $[\alpha], [\beta] \in \mathbb{R}^*$

$$[\alpha] \simeq [\beta] \quad \Leftrightarrow \quad (|\alpha_j - \beta_j| \leq 1/n \quad \text{a.e. for all} \quad n \in \mathbb{N})\ .$$

We mention that for all finite $[\alpha] \in \mathbb{R}^*$ there is an unique real number $r \in \mathbb{R}$ such that $[\alpha] \simeq r$. This real number r of any finite hyperreal number $[\alpha]$ is called the standard part and we write $\mathrm{st}([\alpha]) = r$.

We extend real-valued functions

$$f : \mathbb{R} \to \mathbb{R} : \, x \mapsto f(x)$$

in a canonical way to hyperreal-valued functions.

Definition 5. For $f : \mathbb{R} \to \mathbb{R}$ the ***hyperreal extension*** is

$$f^* : \mathbb{R}^* \to \mathbb{R}^* : \, [\alpha] \mapsto f^*([\alpha]) := [\beta]$$

with $\beta_j := f(\alpha_j)$ for $j \in \mathbb{N}$.

It yields that $f^*([\alpha])$ does not depend on the representative of the equivalence class $[\alpha]$ and for all real numbers $r \in \mathbb{R}$ we have $f^*(r) = f(r)$. In what follows the brackets in the equivalence class for elements of $\mathbb{R}^*$ are omitted. Especially, infinitesimals $[\alpha] \in \mathbb{R}^*$ are indicated by $d\alpha$. Now, differentiability of real-valued functions can be formulated with the use of infinitesimals and the hyperreal extension as follows.

Proposition 3 (Differentiability). *Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0, c \in \mathbb{R}$. Then, the followings are equivalent:*

(i) f is differentiable in x_0 with the derivative $f'(x_0) = c$,
(ii) the limit

$$f'(x_0) := \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists and it yields $f'(x_0) = c$,

(iii) it yields

$$\frac{f^*(x_0+dx)-f(x_0)}{dx} \simeq c$$

for all $0 \neq dx \simeq 0$.

Example 1. We determine the derivative of the function $f(x) = x^2$ in $x_0 \in \mathbb{R}$. It yields for any infinitesimal $0 \neq dx \in \mathbb{R}^*$

$$\frac{f^*(x_0+dx)-f(x_0)}{dx} = \frac{(x_0+dx)^2 - x_0^2}{dx} = 2x_0 + dx \simeq 2x_0$$

since dx is an infinitesimal. This implies that $f'(x_0) = 2x_0$.

Example 2. We derive Euler's number via Nonstandard Analysis. It remains only to show that $\frac{i-1}{i} = 1$ for an infinitely great number i. Then, by defining the infinitely small number $dh := \frac{1}{i}$ we calculate

$$\frac{i-1}{i} = \frac{\frac{1}{dh}-1}{\frac{1}{dh}} = dh \cdot \frac{1-dh}{dh} = 1 - dh \simeq 1\,.$$

Then, it follows as in Euler's work [11, Chapter 7] that

$$a = 1 + \frac{1}{1}k + \frac{1}{2!}k^2 + \frac{1}{3!}k^3 + \frac{1}{4!}k^4 + \cdots + \text{infinitesimal quantities}\,.$$

Since $a \in \mathbb{R}$, it follows that

$$a = 1 + \frac{1}{1}k + \frac{1}{2!}k^2 + \frac{1}{3!}k^3 + \frac{1}{4!}k^4 + \cdots$$

which is for $k = 1$ Euler's number.

2.2 Infinitesimal Fundamentum

We are going to introduce the infinitesimal fundamentum of a real-valued function $f : \mathbb{R} \to \mathbb{R}$. First, we present the ideas of Newton and Leibniz about the derivative of a function. We start with Newton's idea. Newton considers an observed motion $f(t)$ (in Newton's way of speaking a so-called fluent) in a time interval. The main goal is to determine the speed at some time t_0. We can observe the motion at time t_0 and at the time $t_1 > t_0$ then the approximative speed at time t_0 is given by the quotient of the covered distance and the time difference, i.e.

$$\text{speed at time } t_0 \approx \frac{f(t_1)-f(t_0)}{t_1 - t_0}\,,$$

see Figure 1. Newton defines the speed of the motion at time t_0 (in Newton's way of speaking a so-called fluxion) as the quotient

$$\frac{f(t)-f(t_0)}{t-t_0}$$

for an infinitely small time interval, i.e. $t - t_0$ is an infinitesimal, see also [16,15,8].

The investigation of Leibniz is based on the following idea. Leibniz assumes that a curve f depending on the variable x (like the motion f in Newton's context) is an infinite polygon. Then, there is an infinitely small slope triangle such that the tangent at some point x_0 is intersecting the curve in an infinitesimal line, see Figure 2. The slope of the tangent at some point x_0 is given by the quotient

$$\text{slope of tangent} = \frac{df(x_0)}{dx}$$

where dx is an infinitely small number and $df(x_0) = f(x_0+dx) - f(x_0)$, see also [3].

We are going to introduce the infinitesimal fundamentum of a real-valued function $f : \mathbb{R} \to \mathbb{R} : x \mapsto f(x)$. Our approach is based on the ideas of Newton and Leibniz. But we assume that the straight line (linear curve) between the points $(x_0, f(x_0))$ and $(x_0+dx, f(x_0+dx))$ can be replaced in a *suitable way* by an curved line (nonlinear curve). In the coherence of Newton that means we consider a curved line instead of the secant. In the context of Leibniz that means we consider a curved infinitesimal slope triangle, see Figure 2 and Figure 3. We introduce the infinitesimal fundamentum in the following definition.

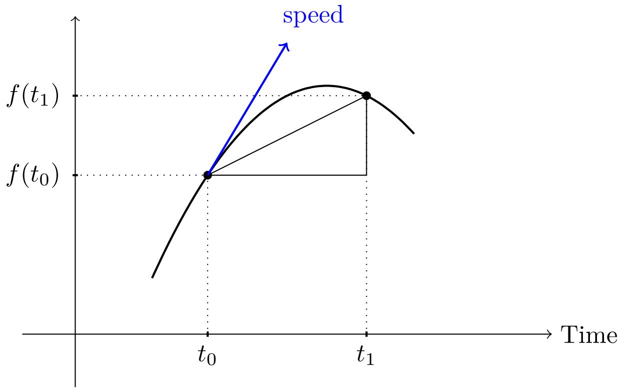

Fig. 1 Newton's idea: Observed motion $f(t)$ at time t_0 and t_1.

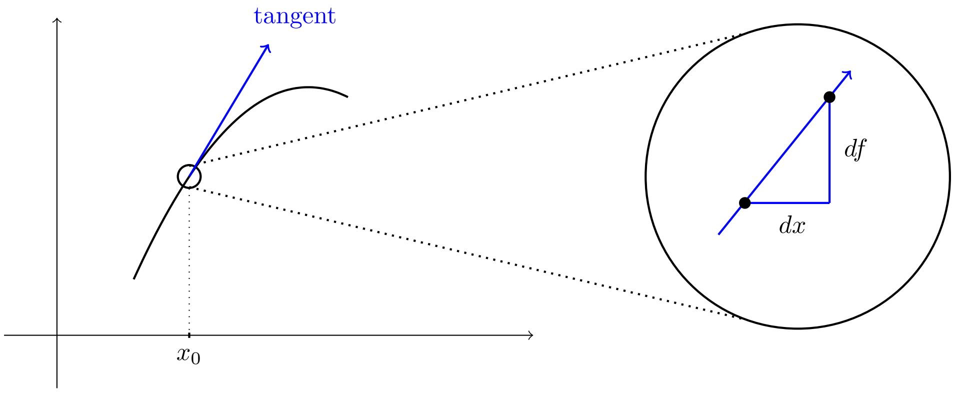

Fig. 2 Leibniz' idea: Infinitesimal slope triangle with infinitesimals df and dx.

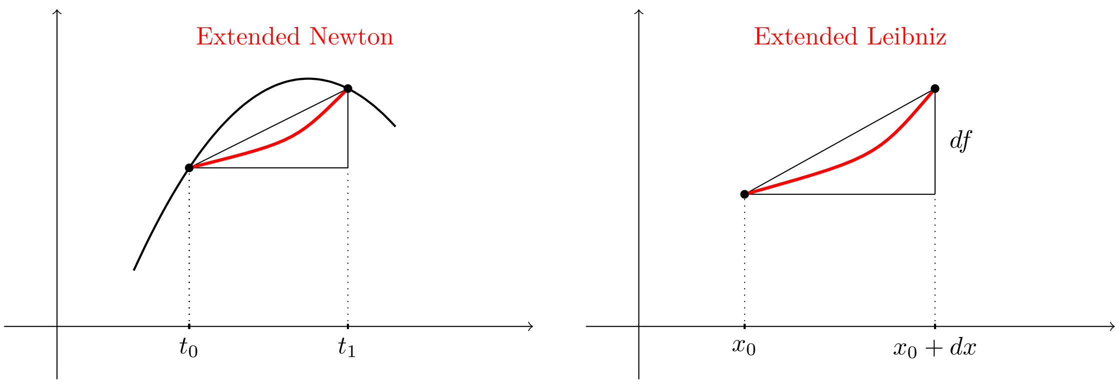

Fig. 3 Idea of the infinitesimal fundametum.

Definition 6. Let $f : \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$ and an infinitesimal $dh \neq 0$ be given. A function $g : \mathbb{R} \to \mathbb{R}$ with

$$\begin{cases} g(x_0) & = f(x_0) \\ g^*(x_0 + dh) & = f^*(x_0 + dh) \end{cases} \tag{3}$$

is called the ***infinitesimal fundamentum*** of f in x_0 with the infinitesimal dh.

Corollary 1. *Let g be the infinitesimal fundamentum of f in $x_0 \in \mathbb{R}$ with the infinitesimal dh. Further, let f and g be differentiable in x_0. Then, it yields*

$$f'(x_0) = g'(x_0)\,.$$

Proof. Using Proposition 3 and (3) we obtain

$$g'(x_0) \simeq \frac{g^*(x_0+dh)-g(x_0)}{dh} = \frac{f^*(x_0+dh)-f(x_0)}{dh} \simeq f'(x_0)$$

for any infinitesimal $0 \neq dh \in \mathbb{R}^*$. Since $f'(x_0), g'(x_0) \in \mathbb{R}$ it yields

$$f'(x_0) = g'(x_0)\,.$$

□

We mention that the choice of the infinitesimal fundamentum is arbitrary and that the infinitesimal fundamentum which is used in Newton's and Leibniz' context is given by a linear infinitesimal fundamentum, see the following Definition.

Definition 7. We call the function

$$g^*(z) = \frac{f^*(x_0+dh)-f(x_0)}{dh}(z-x_0)+f(x_0) \tag{4}$$

the ***linear infinitesimal fundamentum*** of f in x_0 with the infinitesimal dh.

Next, we state the exponential infinitesimal fundamentum which will be studied in detail in the next section. In this case we do not have a straight line in the infinitesimal slope triangle.

Definition 8. We call the function

$$g^*(z) = \frac{f^*(x_0+dh)-f(x_0)}{e^{dh}-1}\left(e^{z-x_0}-1\right)+f(x_0) \tag{5}$$

the ***exponential infinitesimal fundamentum*** of f in x_0 with the infinitesimal dh.

The exponential infinitesimal fundamentum of differentiable functions f yields by using Corollary 1 and Proposition 3 the equality

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{e^h-1} \tag{6}$$

for any $x \in \mathbb{R}$. Note that equation (6) is verified for continuously differentiable functions f by using l'Hospital's rule.

Remark 2. We mention that the exponential infinitesimal fundamentum (5) is a smooth function and we obtain a representation of higher order derivatives of the order $n \in \mathbb{N}$ through

$$f^{(n)}(x_0) \simeq \frac{d^{(n)}f(x_0)}{\sum\limits_{k=0}^{n}\binom{n}{k}(-1)^k e^{kdh}}$$

where we have defined the operator $df(x_0) := f^*(x_0+dh)-f(x_0)$ for smooth functions f in $x_0 \in \mathbb{R}$ with the infinitesimal dh.

2.3 Examples For Infinitesimal Fundamenta

We list examples for several infinitesimal fundamenta and consequential the equivalent formulation for the derivative of a real-valued function $f: \mathbb{R} \to \mathbb{R}$ in $x_0 \in \mathbb{R}$.

1. The standard infinitesimal fundamentum as it was used by Newton and Leibniz is the linear infinitesimal fundamentum and given by

$$g^*(z) = \frac{f^*(x_0+dh)-f(x_0)}{dh}(z-x_0)+f(x_0)$$

   with the infinitesimal dh. The resulting formulation of the derivative is the well-known formula

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\,.$$

2. We consider the quadratic infinitesimal fundamentum

$$g^*(z) = az^2$$

for some $a \in \mathbb{R}^*$. It yields for $x_0 \neq 0$

$$2ax_0 \simeq g'(x_0) = f'(x_0) \tag{7}$$

and for any infinitesimal dh

$$g^*(x_0 + dh) - g(x_0) = a(x_0 + dh)^2 - ax_0^2 = 2ax_0 dh + adh^2 = f^*(x_0 + dh) - f(x_0)\,.$$

This implies by using (7)

$$f'(x_0) \simeq \frac{f^*(x_0 + dh) - f(x_0)}{dh} - adh\,.$$

It remains to specify a. Using $a = \frac{f^*(x_0+dh)-f(x_0)}{2x_0 dh+dh^2}$ results in

$$f'(x_0) \simeq \frac{f^*(x_0 + dh) - f(x_0)}{dh} - \frac{f^*(x_0 + dh) - f(x_0)}{2x_0 + dh} \simeq \frac{f^*(x_0 + dh) - f(x_0)}{dh}$$

since $\frac{f^*(x_0+dh)-f(x_0)}{2x_0+dh}$ is an infinitesimal. We obtain as the for the linear infinitesimal fundamentum the relation

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}\,.$$

3. We state a modified exponential infinitesimal fundamentum of the form

$$g^*(z) = \frac{f^*(x_0 + dh) - f(x_0)}{e^{\lambda dh} - 1}\left(e^{\lambda(z-x_0)} - 1\right) + f(x_0)$$

for $\lambda \in \mathbb{R} \setminus \{0\}$. The resulting derivative is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{\frac{e^{\lambda dh}-1}{\lambda}}\,.$$

4. We choose an infinitesimal fundamentum of the form

$$g^*(z) = a\sin(z) + b \tag{8}$$

for some $a, b \in \mathbb{R}^*$. It yields

$$g'(x_0) \simeq a\cos(x_0)$$

and for any infinitesimal dh

$$g^*(x_0 + dh) - g(x_0) = a\sin(x_0 + dh) - a\sin(x_0)\,.$$

Addition theorems imply

$$g^*(x_0 + dh) - g(x_0) = a(\sin(x_0)\cos(dh) + \sin(dh)\cos(x_0)) - \sin(x_0))$$

and so

$$g^*(x_0 + dh) - g(x_0) = a(\sin(x_0)(\cos(dh) - 1) + \sin(dh)\cos(x_0)) = f^*(x_0 + dh) - f(x_0)\,.$$

Using $f(x_0) = g(x_0) = a\sin(x_0) + b$ it yields

$$f^*(x_0 + dh) - f(x_0) - (f(x_0) - b)(\cos(dh) - 1) = a\cos(x_0)\sin(dh)\,.$$

We obtain by using

$$f'(x_0) = g'(x_0) \simeq a\cos(x_0)$$

following relation

$$f'(x_0) \simeq \frac{f(x_0 + dh) - \cos(dh)f(x_0) + b(\cos(dh) - 1)}{\sin(dh)} = \frac{f(x_0 + dh) - \cos(dh)f(x_0)}{\sin(dh)} + b\,\frac{\cos(dh) - 1}{\sin(dh)}\,.$$

Altogether, we have

$$f'(x_0) \simeq \frac{f(x_0+dh) - \cos(dh) f(x_0)}{\sin(dh)}$$

since $b\frac{\cos(dh)-1}{\sin(dh)}$ is an infinitesimal. In the limit notation that means

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - \cos(h) f(x_0)}{\sin(h)}\,.$$

5. An infinitesimal fundamentum of the form

$$g^*(z) = a\cos(z) + b$$

for some $a, b \in \mathbb{R}^*$ yields as for the sine the same relation

$$f'(x_0) \simeq \frac{f(x_0+dh) - \cos(dh) f(x_0)}{\sin(dh)}\,.$$

6. We consider a logarithmic infinitesimal fundamentum of the form

$$g^*(z) = a\ln(z) + b$$

for some $a, b, z \in \mathbb{R}^*$ with $z > 0$. It yields for all $x_0 > 0$

$$\frac{a}{x_0} \simeq g'(x_0) = f'(x_0)$$

and for any infinitesimal dh

$$g^*(x_0+dh) - g(x_0) = a\ln(x_0+dh) - a\ln(x_0) = a\ln\left(1 + \frac{dh}{x_0}\right) = f^*(x_0+dh) - f(x_0)\,.$$

We obtain by using $a \simeq x_0\, f'(x_0)$

$$f'(x_0) \simeq \frac{f^*(x_0+dh) - f(x_0)}{x_0\, \ln\left(1+\frac{dh}{x_0}\right)}\,.$$

This implies for all $x_0 > 0$

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{x_0\, \ln\left(1+\frac{h}{x_0}\right)}\,.$$

7. We fix a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $x_0 \in \mathbb{R}^+$ and choose an infinitesimal fundamentum of the form

$$g^*(z) = a\sqrt{z}$$

for some $a \in \mathbb{R}^*$. It yields for all $x_0 > 0$

$$\frac{a}{2\sqrt{x_0}} \simeq g'(x_0) = f'(x_0)$$

and for any infinitesimal dh

$$(g^*(x_0+dh))^2 - (g(x_0))^2 = a^2(x_0+dh) - a^2 x_0 = a^2 dh = (f^*(x_0+dh))^2 - (f(x_0))^2\,.$$

Using

$$a^2 \simeq 2a\sqrt{x_0} f'(x_0) = 2f(x_0) f'(x_0)$$

we obtain

$$f'(x_0) \simeq \frac{(f^*(x_0+dh))^2 - (f(x_0))^2}{2f(x_0)dh} = \frac{f^*(x_0+dh) + f(x_0)}{2f(x_0)} \cdot \frac{f^*(x_0+dh) - f(x_0)}{dh}\,.$$

That means

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) + f(x_0)}{2f(x_0)} \cdot \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}\,.$$

3 Expansion Formula

We introduce Taylor's expansion formula which can be obtained by the linear infinitesimal fundamentum (4). Using the linear infinitesimal fundamentum the derivative at $x \in \mathbb{R}$ is given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\,.$$

For $f \in C([a,b],\mathbb{R})$ and f differentiable in (a,b) the mean value theorem implies some $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'(c)$$

or equivalently

$$\forall x \in [a,b],\ \forall h \in \mathbb{R},\ |h| < \min\{|x-a|, |b-x|\},\ \exists c \in (a,b) :\ f(x+h) = f(x) + hf'(c)\,.$$

Taylor's expansion formula is a series expansion of functions of the form

$$f(x+h) = f(x) + hf'(x) + h^2\frac{f''(x)}{2!} + h^3\frac{f^{(3)}(x)}{3!} + \cdots \tag{9}$$

which a polynomial in the variable h with coefficients depending on the derivatives of the considered function f. The goal is to obtain a new Taylor's expansion formula obtained by an exponential infinitesimal fundamentum (5), i.e. we search a representation of the function f at some point $x+h$ of the form

$$f(x+h) = c_0(x) + c_1(x)(e^h-1) + c_2(x)(e^h-1)^2 + c_3(x)(e^h-1)^3 + \cdots$$

for some coefficients $c_0(x), c_1(x), c_2(x), \ldots$ depending on the derivatives of f. The main result in this section is given by the following theorem.

Theorem 1. *Let $a,b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}_0$. Further, let $f \in C^n([a,b])$ and $f^{(n)}$ differentiable in (a,b). Then, there is a constant $c \in (a,b)$ such that*

$$f(b) = f(a) + \sum_{j=1}^{n} \frac{(e^{b-a}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(a) + R_n(b,a,c)\,, \tag{10}$$

where

$$R_n(b,a,c) = e^{c-b}\frac{(e^{b-a}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(c)$$

and $j,k \in \mathbb{N}$ with $k \le j$

$$a_k^j := \begin{cases} 1 & \textit{for} \quad k = 1, j \ge 1\,, \\ -(j-1)a_{k-1}^{j-1} + a_k^{j-1} & \textit{for} \quad 1 < k < j\,, \\ (-1)^{j+1}(j-1)! & \textit{for} \quad j = k \ge 1\,. \end{cases} \tag{11}$$

Corollary 2. *Let $a,b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$. Further, let $f \in C^n([a,b])$ and $f^{(n)}$ differentiable in (a,b). Then, the following holds:*
$\forall x \in [a,b],\ \forall h \in \mathbb{R},\ |h| < \min\{|x-a|, |b-x|\},\ \exists \zeta \in (a,b)$:

$$f(x+h) = f(x) + \sum_{j=1}^{n} \frac{(e^h-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(x) + R_n(h+x,x,\zeta)\,, \tag{12}$$

where

$$R_n(h+x,x,\zeta) = e^{\zeta-x-h}\frac{(e^h-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(\zeta)\,. \tag{13}$$

First, we illustrate in Figure 4 an explicit representation of the coefficients a_k^j used in (10) and (12). In Figure 5 we state an algorithm to calculate the coefficients a_k^{j+1} for given coefficients a_k^j. Note that the sum over the coefficients is equal zero for all $n \in \mathbb{N}$, i.e.

$$\sum_{k=1}^{n} a_k^n = 0\,.$$

In the case that the rest term (13) vanishes for a smooth function $f \in C^\infty([a,b])$ we obtain an expansion formula of the form

$$f(x+h) = f(x) + f'(x)(e^h - 1) + \frac{f''(x) - f'(x)}{2!}(e^h-1)^2 + \frac{f'''(x) - 3f''(x) + 2f'(x)}{3!}(e^h-1)^3 + \cdots\,,$$

where $x + h \in [a,b]$.

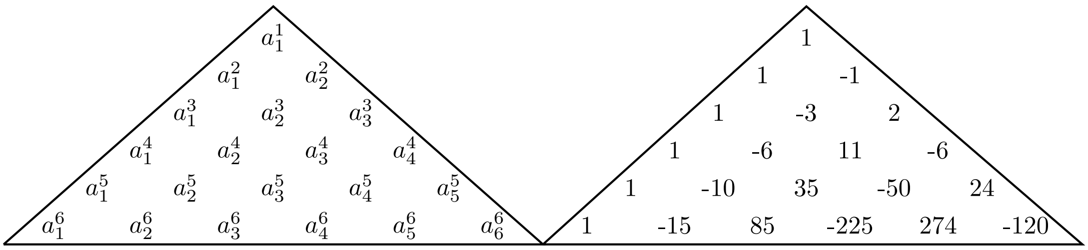

Fig. 4 Coefficients a_k^j (see (11)) in the expansion formula (10) up to the order $n = 6$.

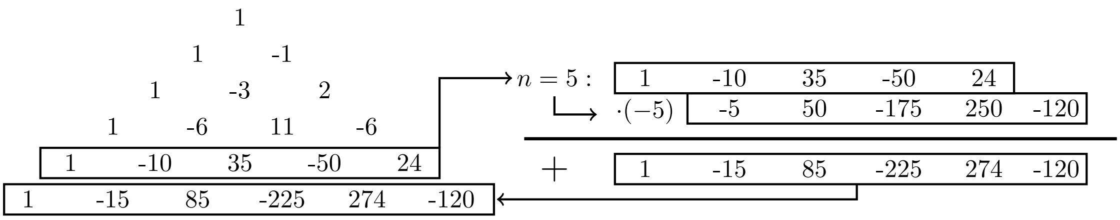

Fig. 5 Algorithm for determination the coefficients a_k^6 from the previous coefficients a_k^5.

3.1 Proof Of Theorem 1

We are going to prove Theorem 1. For illustrating we start proving the theorem for the trivial case $n = 1$. Then, the general case $n \in \mathbb{N}$ will be proved. For the case $n = 1$ we have to show that there is some $c \in (a,b)$ such that

$$f(b) = f(a) + (e^{b-a} - 1)f'(a) + e^{c-b}\frac{(e^{b-a}-1)^2}{2!}(f''(c) - f'(c)) \tag{14}$$

We define

$$F(t) := f(b) - f(t) - (e^{b-t} - 1)f'(t) - m\frac{(e^{b-t}-1)^2}{2!}$$

with

$$m := \frac{f(b) - f(a) - (e^{b-a}-1)f'(a)}{\frac{(e^{b-a}-1)^2}{2!}}\,.$$

Then, it yields

$$F(b) = 0 = F(a)$$

such that the mean value theorem implies some $c \in (a, b)$ with $F'(c) = 0$, i.e.

$$0 = F'(c) = -f'(c) - (e^{b-c} - 1)f''(c) + e^{b-c}f'(c) + m(e^{b-c} - 1)e^{b-c}\,.$$

We obtain

$$0 = F'(c) = (e^{b-c} - 1)\left((f'(c) - f''(c)) + me^{b-c}\right)$$

and so

$$m = (f''(c) - f'(c))e^{c-b}\,.$$

Using $F(a) = 0$ we obtain

$$f(b) = f(a) + (e^{b-a} - 1)f'(a) + e^{c-b}\frac{(e^{b-a} - 1)^2}{2!}(f''(c) - f'(c))$$

as stated in (14).

Proof of Theorem 1 for the general case $n \in \mathbb{N}$. We define

$$F(t) := f(b) - f(t) - \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) - m\frac{(e^{b-t} - 1)^{n+1}}{(n + 1)!}$$

with

$$m := \frac{f(b) - f(a) - \sum\limits_{j=1}^{n} \frac{(e^{b-a}-1)^j}{j!} \sum\limits_{k=1}^{j} a_k^j f^{(j+1-k)}(a)}{\frac{(e^{b-a}-1)^{n+1}}{(n+1)!}}\,.$$

The function F satisfies $F(b) = 0 = F(a)$ and the mean value theorem yields some $c \in (a, b)$ with $0 = F'(c)$. We determine the derivative

$$\begin{aligned} F'(t) = -f'(t) &- \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) \\ &+e^{b-t} \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^{j-1}}{(j - 1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) + me^{b-t}\frac{(e^{b-t} - 1)^n}{n!}\,. \end{aligned}$$

First, we claim that for all $n \in \mathbb{N}$

$$\begin{aligned} -f'(t) - \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^{j-1}}{(j - 1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\ = -\frac{(e^{b-t} - 1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t) \end{aligned} \tag{15}$$

which will be proved with the help of an induction.
Induction Basis $n = 1$: We obtain

$$\begin{aligned} -f'(t) - (e^{b-t} - 1)a_1^1 f''(t) + e^{b-t}a_1^1 f'(t) &= -(e^{b-t} - 1)(f''(t) - f'(t)) \\ &= -(e^{b-t} - 1)(a_1^2 f''(t) + a_2^2 f'(t)) \end{aligned}$$

which shows that the statement (15) holds for $n = 1$.
Induction Hypothesis: We assume that the statement (15) holds for $n \in \mathbb{N}$, i.e.

$$\begin{aligned} -f'(t) - \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n} \frac{(e^{b-t} - 1)^{j-1}}{(j - 1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\ = -\frac{(e^{b-t} - 1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t)\,. \end{aligned} \tag{16}$$

Then, we show under the hypothesis (16) that statement (15) is satisfied for $n+1$.
Inductive Step $n \rightsquigarrow n+1$**:** We calculate

$$\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -f'(t) - \sum_{j=1}^{n} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&\quad - \frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+3-k)}(t) + e^{b-t} \frac{(e^{b-t}-1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t) \,.
\end{aligned}$$

Using the induction hypothesis (16) yields

$$\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t) - \frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+3-k)}(t) \qquad (17) \\
&\quad + e^{b-t} \frac{(e^{b-t}-1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t) \,.
\end{aligned}$$

We take on the right hand-side in (17) the first and last term together. Further, we factor out $-\frac{(e^{b-t}-1)^{n+1}}{(n+1)!}$ which implies

$$\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(\sum_{k=1}^{n+1} a_k^{n+1} f^{(n+3-k)}(t) - (n+1) a_k^{n+1} f^{(n+2-k)}(t) \right) .
\end{aligned}$$

A shifting in the index yields

$$\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(\sum_{k=0}^{n} a_{k+1}^{n+1} f^{(n+2-k)}(t) - \sum_{k=1}^{n+1} (n+1) a_k^{n+1} f^{(n+2-k)}(t) \right) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(\sum_{k=1}^{n} a_{k+1}^{n+1} f^{(n+2-k)}(t) - \sum_{k=1}^{n} (n+1) a_k^{n+1} f^{(n+2-k)}(t) \right) \\
&\quad - \frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(a_1^{n+1} f^{(n+2)}(t) - (n+1) a_{n+1}^{n+1} f'(t) \right) .
\end{aligned}$$

We obtain

$$\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(a_1^{n+1} f^{(n+2)}(t) - (n+1) a_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n} f^{(n+2-k)}(t) \big(a_{k+1}^{n+1} - (n+1) a_k^{n+1} \big) \right) .
\end{aligned}$$

Using

$$\big(a_{k+1}^{n+1} - (n+1) a_k^{n+1} \big) = a_{k+1}^{n+2}$$

and shifting again in the index yields

$$
\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(a_1^{n+1} f^{(n+2)}(t) - (n+1) a_{n+1}^{n+1} f'(t) + \sum_{k=2}^{n+1} a_k^{n+2} f^{(n+3-k)}(t) \right) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(a_1^{n+1} f^{(n+2)}(t) - (n+1) a_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n+1} a_k^{n+2} f^{(n+3-k)}(t) - a_1^{n+2} f^{(n+2)}(t) \right) .
\end{aligned}
$$

Using $a_1^{n+1} = a_1^{n+2}$ and changing the limit in the sum yields

$$
\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \left(-(n+1) a_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n+2} a_k^{n+2} f^{(n+3-k)}(t) - a_{n+2}^{n+2} f'(t) \right) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} a_k^{n+2} f^{(n+3-k)}(t) + \frac{(e^{b-t}-1)^{n+1}}{(n+1)!} f'(t) \big((n+1) a_{n+1}^{n+1} + a_{n+2}^{n+2} \big) .
\end{aligned}
$$

Using $a_{n+1}^{n+1} = (-1)^{n+2}\, n!$ and $a_{n+2}^{n+2} = (-1)^{n+3} (n+1)!$ yields

$$
\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} a_k^{n+2} f^{(n+3-k)}(t) + \frac{(e^{b-t}-1)^{n+1}}{(n+1)!} f'(t) (n+1)! (-1)^{n+2} \big(1 + (-1) \big) .
\end{aligned}
$$

End of Induction: Altogether, we obtain

$$
\begin{aligned}
-f'(t) &- \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) + e^{b-t} \sum_{j=1}^{n+1} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) \\
&= -\frac{(e^{b-t}-1)^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} a_k^{n+2} f^{(n+3-k)}(t)
\end{aligned}
$$

which completes the induction and we obtain (15) for all $n \in \mathbb{N}$. That means for the derivative of F

$$
\begin{aligned}
F'(t) &= -f'(t) - \sum_{j=1}^{n} \frac{(e^{b-t}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+2-k)}(t) \\
&\quad + e^{b-t} \sum_{j=1}^{n} \frac{(e^{b-t}-1)^{j-1}}{(j-1)!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(t) + m e^{b-t} \frac{(e^{b-t}-1)^n}{n!} \\
&= -\frac{(e^{b-t}-1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(t) + m e^{b-t} \frac{(e^{b-t}-1)^n}{n!} .
\end{aligned}
$$

The mean value theorem yields some $c \in (a, b)$ with $F'(c) = 0$ which means

$$
0 = F'(c) = -\frac{(e^{b-c}-1)^n}{n!} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(c) + m e^{b-c} \frac{(e^{b-c}-1)^n}{n!} . \tag{18}
$$

Equation (18) implies

$$m = e^{c-b} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(c)\,.$$

Using $F(a) = 0$ we obtain

$$f(b) = f(a) + \sum_{j=1}^{n} \frac{(e^{b-a}-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(a) + e^{c-b} \sum_{k=1}^{n+1} a_k^{n+1} f^{(n+2-k)}(c) \frac{(e^{b-a}-1)^{n+1}}{(n+1)!}$$

which completes the proof of Theorem 1. □

We remark that as for the exponential infinitesimal fundamentum we can derive the following for the sine case.

Proposition 4. *Let $[a,b] \subset (0,\pi)$. Further, let $f \in C([a,b])$ and f differentiable in (a,b). Then, there is a constant $c \in (a,b)$ such that*

$$f(b) = \cos(b-a) f(a) + \sin(b-a) \cdot (f'(c) + \tan(b-c) f(c))\,.$$

Proof. Consider

$$F(t) := f(b) - \cos(b-t) f(t) - \sin(b-t) m$$

with

$$m := \frac{f(b) - \cos(b-a) f(a)}{\sin(b-a)}\,.$$

Then, it yields $F(b) = 0 = F(a)$. The mean value theorem implies some $c \in (a,b)$ such that $F'(c) = 0$. We obtain

$$0 = F'(c) = -\cos(b-c) f'(c) - \sin(b-c) f(c) + m \cos(b-c)$$

which yields

$$m = (f'(c) + \tan(b-c) f(c))\,.$$

Using $F(a) = 0$ implies

$$f(b) = \cos(b-a) f(a) + \sin(b-a) \cdot (f'(c) + \tan(b-c) f(c))$$

as stated in the proposition. □

Next, we compare Taylor's expansion formula (9) with the expansion formula (12) which is obtained by an exponential infinitesimal fundamentum (4) for some examples. For a smooth function f we consider the expansions

$$f(x+h) = f(x) + f'(x) h + \frac{f''(x)}{2!} h^2 + \frac{f^{(3)}(x)}{3!} h^3 + \cdots$$

and

$$f(x+h) = f(x) + f'(x)(e^h - 1) + \frac{f''(x) - f'(x)}{2!}(e^h-1)^2 + \frac{f^{(3)}(x) - 3f''(x) + 2f'(x)}{3!}(e^h-1)^3 + \cdots$$

up to a given order for some $x \in \mathbb{R}$ and $h \in \mathbb{R}$.

Example 3. We consider $f(x) = e^{\sin(5x)}$ on the interval $[-1, 1/3]$. Then, we calculate the expansions of second order at the expansion point $x_0 = 0$

$$f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 = 1 + 5x + \frac{25}{2} x^2$$

and

$$f(0) + f'(0)(e^x - 1) + \frac{f''(0) - f'(0)}{2!}(e^x-1)^2 = 1 + 5(e^x - 1) + 10(e^x-1)^2\,,$$

see Figure 6 for the plots.

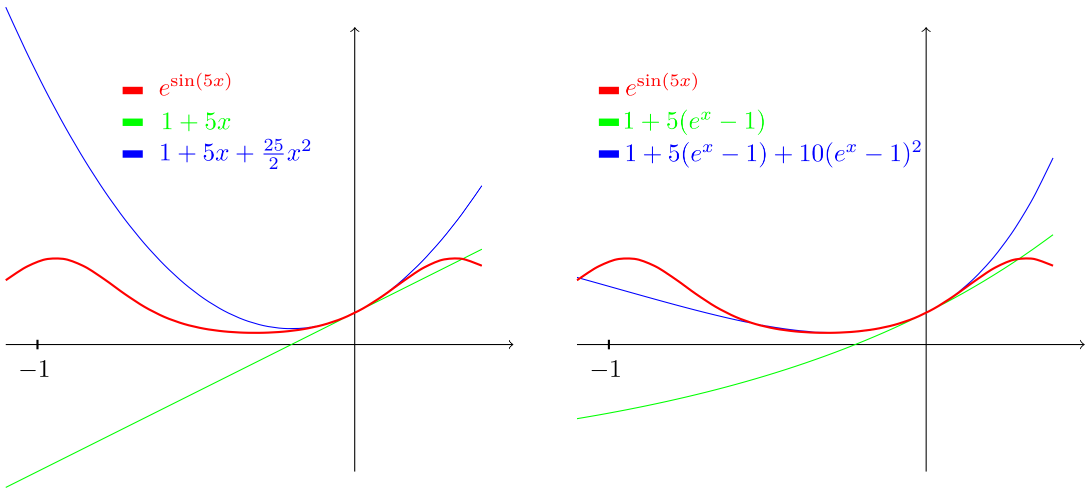

Fig. 6 Plots for $f(x) = \exp(\sin(5x))$ on the interval $[-1, 1/3]$ and the approximations of first and second order resulting from the expansion formulae (9) and (12). The approximation on the right resulting from (12) is closer to the considered function $f(x) = \exp(\sin(5x))$ in the first and in the second order.

Example 4. We consider $f(x) = \frac{1}{1-x}$ on the interval $[0, 8/10]$ and the expansion point $x_0 = 0$. We obtain via Taylor's formula (9)

$$T_6(x) := \sum_{j=0}^{6} \frac{f^{(j)}(0)}{j!} x^j = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$$

and via the expansion formula (12)

$$\begin{aligned} E_6(x) &:= 1 + \sum_{j=1}^{6} \frac{(e^x-1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(0) \\ &= 1 + (e^x - 1) + \frac{1}{2}(e^x-1)^2 + \frac{1}{3}(e^x-1)^3 + \frac{1}{6}(e^x-1)^4 + \frac{7}{60}(e^x-1)^5 + \frac{19}{360}(e^x-1)^6 \end{aligned}$$

see Figure 7 for the plots and the comparison of the approximations.

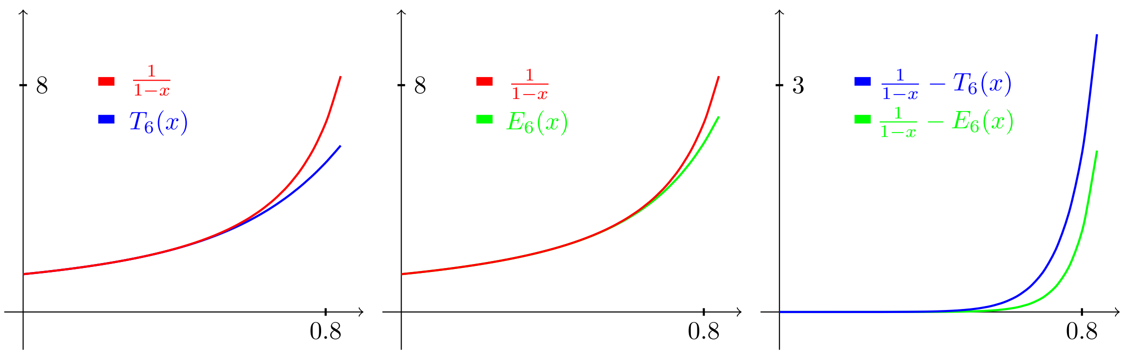

Fig. 7 Plots of the approximations for $f(x) = \frac{1}{1-x}$ of sixth order resulting from the expansion formulae (9) and (12). The error functions on the right graph show that the approximation resulting from (12) is closer to the considered function f where $T_6(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ and $E_6(x) = 1 + (e^x-1) + \frac{1}{2}(e^x-1)^2 + \frac{1}{3}(e^x-1)^3 + \frac{1}{6}(e^x-1)^4 + \frac{7}{60}(e^x-1)^5 + \frac{19}{360}(e^x-1)^6$.

3.2 Expansion Formula Via Slow Exponential Infinitesimal Fundamentum

The goal is to derive another new expansion formula resulting from the infinitesimal fundamentum

$$g^*(z) = \frac{f^*(x_0 + dh) - f(x_0)}{1 - e^{-dh}} \left(1 - e^{x_0 - z}\right) + f(x_0) \tag{19}$$

which we call the ***slow*** exponential infinitesimal fundamentum. That means we search an expansion for the smooth function f of the form

$$f(x+h) = c_0(x) + c_1(x)(1 - e^{-h}) + c_2(x)(1 - e^{-h})^2 + c_3(x)(1 - e^{-h})^3 + \cdots$$

for some coefficients $c_0(x), c_1(x), c_2(x), ...$ depending on the derivatives of f. The main result in this issue is given by the following theorem.

Theorem 2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}_0$. Further, let $f \in C^n([a, b])$ and $f^{(n)}$ differentiable in (a, b). Then, there is a constant $c \in (a, b)$ such that*

$$f(b) = f(a) + \sum_{j=1}^{n} \frac{(1 - e^{a-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(a) + R_n(b, a, c)\,, \tag{20}$$

where

$$R_n(b, a, c) = e^{b-c} \frac{(1 - e^{a-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(c)$$

and $j, k \in \mathbb{N}$ with $k \leq j$

$$b_k^j := \begin{cases} 1 & \textit{for} \quad k = 1, j \geq 1\,, \\ (j-1) b_{k-1}^{j-1} + b_k^{j-1} & \textit{for} \quad 1 < k < j\,, \\ (j-1)! & \textit{for} \quad j = k \geq 1\,. \end{cases} \tag{21}$$

Proof. The proof of Theorem 2 can be handled as the proof of Theorem 1. We omit the proof at this point and move it to Appendix A. □

A simple consequence is given by the following corollary.

Corollary 3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $n \in \mathbb{N}$. Further, let $f \in C^n([a, b])$ and $f^{(n)}$ differentiable in (a, b). Then, the following holds:*
$\forall x \in [a, b]$, $\forall h \in \mathbb{R}$, $|h| < \min\{|x - a|, |b - x|\}$, $\exists \zeta \in (a, b)$:

$$f(x+h) = f(x) + \sum_{j=1}^{n} \frac{(1 - e^{-h})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(x) + R_n(h + x, x, \zeta)\,, \tag{22}$$

where

$$R_n(h + x, x, \zeta) = e^{x+h-\zeta} \frac{(1 - e^{-h})^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(\zeta)\,.$$

Note that the relation of the coefficients b_k^j to the coefficients a_k^j is given by $b_k^j = |a_k^j|$. In the following we present Figure 8 and Figure 9 where one can see an overview of the coefficients b_k^j and an algorithm for calculating the coefficients.

Altogether, we consider for a smooth function f the expansions

$$\begin{aligned} f(x+h) &= f(x) + f'(x)h + \frac{f''(x)}{2!} h^2 + \cdots\,, \\ f(x+h) &= f(x) + f'(x)(e^h - 1) + \frac{f''(x) - f'(x)}{2!} (e^h - 1)^2 + \cdots\,, \\ f(x+h) &= f(x) + f'(x)(1 - e^{-h}) + \frac{f''(x) + f'(x)}{2!} (1 - e^{-h})^2 + \cdots \end{aligned}$$

for some $x \in \mathbb{R}$ and $h \geq 0$.

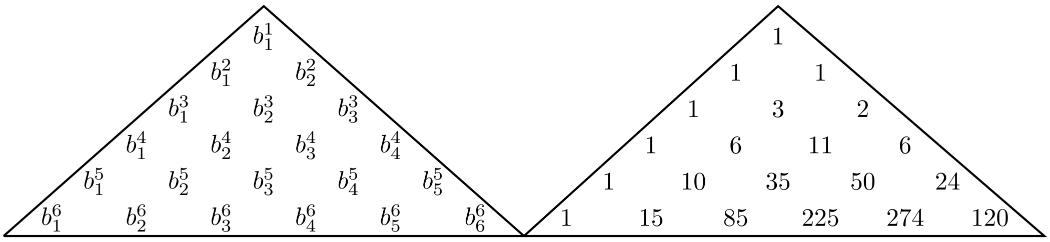

Fig. 8 Coefficients b_k^j (see (21)) in the expansion formula (20) up to the order $n = 6$.

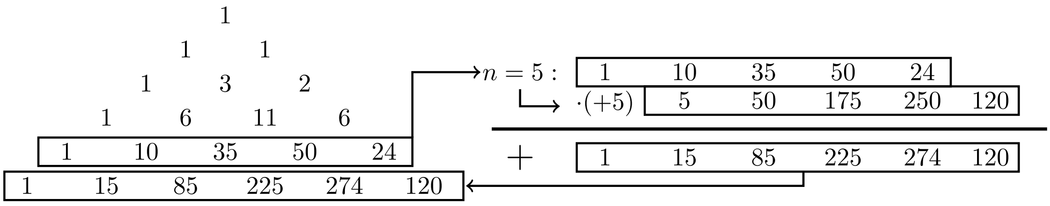

Fig. 9 Algorithm for determination the coefficients b_k^6 from the previous coefficients b_k^5.

Example 5. We consider again $f(x) = \frac{1}{1-x}$ on the interval $[0, 8/10]$ and the expansion point $x_0 = 0$. We obtain via Taylor's formula (9)

$$T_6(x) := \sum_{j=0}^{6} \frac{f^{(j)}(0)}{j!} x^j = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$$

and via the expansion formula (12)

$$\begin{aligned} E_6(x) &:= 1 + \sum_{j=1}^{6} \frac{(e^x - 1)^j}{j!} \sum_{k=1}^{j} a_k^j f^{(j+1-k)}(0) \\ &= 1 + (e^x - 1) + \frac{1}{2}(e^x - 1)^2 + \frac{1}{3}(e^x - 1)^3 + \frac{1}{6}(e^x - 1)^4 + \frac{7}{60}(e^x - 1)^5 + \frac{19}{360}(e^x - 1)^6 \end{aligned}$$

and via the expansion formula (22)

$$\begin{aligned} D_6(x) &:= 1 + \sum_{j=1}^{6} \frac{(1 - e^{-x})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(0) \\ &= 1 + (1 - e^{-x}) + \frac{3}{2}(1 - e^{-x})^2 + \frac{7}{3}(1 - e^{-x})^3 + \frac{11}{3}(1 - e^{-x})^4 + \frac{347}{60}(1 - e^{-x})^5 + \frac{3289}{360}(1 - e^{-x})^6 \end{aligned}$$

see Figure 10 for the plots.

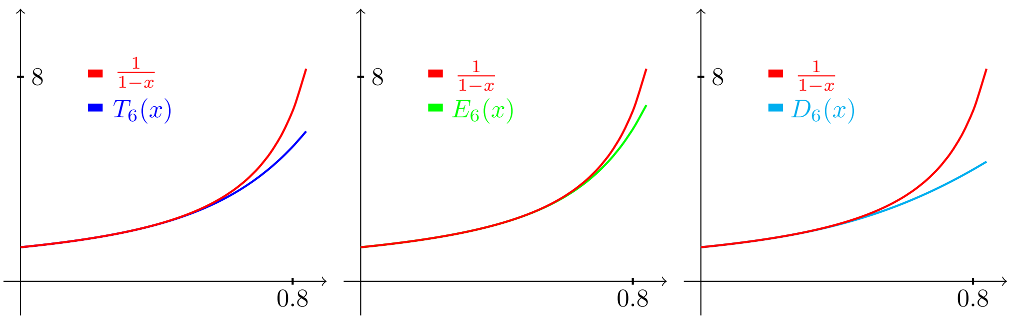

Fig. 10 Plots for $f(x) = \frac{1}{1-x}$ and its approximations of sixth order resulting from the expansion formulae (9), (12) and (22).

4 Applications

In this section we present some applications which can be used in analysis, mathematical physics and many other scientific disciplines. The new kind of definition of the derivative affects a broad field.

4.1 Approximation Formulae

One of the most important application of expansions is the resulting approximation formula of first order around the expansion point. For $|h| \ll 1$ we approximate a function $f : \mathbb{R} \to \mathbb{R}$ at the point $x_0 + h$ by using the linear infinitesimal fundamentum (see (4)) through

$$f(x_0 + h) \approx f(x_0) + f'(x_0)h =: T_1(h, x_0, f)$$

where x_0 is the given expansion point. In the same way we obtain the approximation

$$f(x_0 + h) \approx f(x_0) + f'(x_0)(e^h - 1) =: E_1(h, x_0, f)$$

with the use of an exponential infinitesimal fundamentum (see (5)),

$$f(x_0 + h) \approx f(x_0) + f'(x_0)(1 - e^{-h}) =: D_1(h, x_0, f)$$

with the use of a slow exponential infinitesimal fundamentum (see (19)) and

$$f(x_0 + h) \approx \cos(h)f(x_0) + f'(x_0)\sin(h) =: S_1(h, x_0, f)$$

with a sine infinitesimal fundamentum (see (8)), see Figure 11 for a sketch.

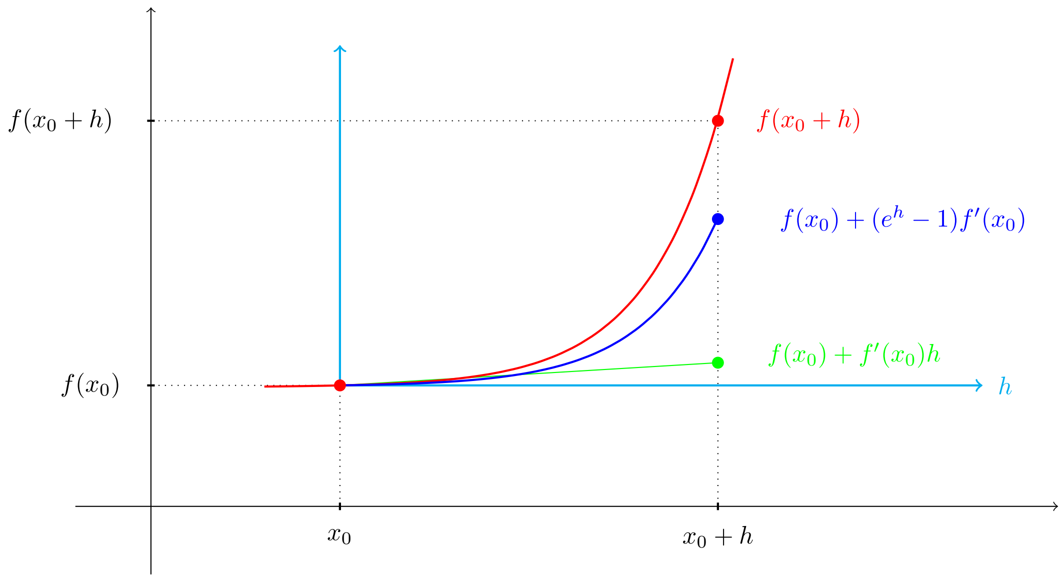

Fig. 11 Approximation Formulae resulting from different infinitesimal fundamenta.

Example 6. For the increasing function $f(x) = e^{x^2}$ we obtain $T_1(h, 1, f) = e + 2eh$, $E_1(h, 1, f) = e + 2e(e^h - 1)$, $D_1(h, 1, f) = e + 2e(1 - e^{-h})$ and $S_1(h, 1, f) = \cos(h)e + 2e\sin(h)$ at the expansion point $x_0 = 1$. We calculate for several values h the functions f, T_1, E_1, D_1 and S_1, see Table 1.

$x_0 = 1$	$h = 0$	$h = 0.005$	$h = 0.1$	$h = 0.5$	$h = 1$
$f(1+h)$	**2.71828**	**2.74567**	**3.35348**	**9.48774**	**54.5982**
$T_1(h)$	2.71828	2.74546	3.26194	5.43656	8.15485
$E_1(h)$	2.71828	2.74553	3.29005	6.24510	12.0598
$D_1(h)$	2.71828	2.74540	3.23564	4.85740	6.15485
$S_1(h)$	2.71828	2.74543	3.24745	4.99194	6.04340

Table 1 $f(x) = e^{x^2}$, $T_1(h,1,f) = e + 2eh$, $E_1(h,1,f) = e + 2e(e^h - 1)$, $D_1(h,1,f) = e + 2e(1 - e^{-h})$ and $S_1(h,1,f) = \cos(h)e + 2e\sin(h)$.

Example 7. We consider the decreasing function $g(x) = \frac{1}{\sqrt{1+x}}$ on the interval $[0,1]$ and the expansion point $x_0 = 0$. We obtain $T_1(h,0,g) = 1-1/2h$, $E_1(h,0,g) = 1-1/2(e^h-1)$, $D_1(h,0,g) = 1-1/2(1-e^{-h})$ and $S_1(h,0,g) = \cos(h) - 1/2\sin(h)$. We calculate for several values h the functions f, T_1, E_1, D_1 and S_1, see Table 2.

$x_0 = 0$	$h = 0$	$h = 0.005$	$h = 0.1$	$h = 0.5$	$h = 1$
$g(h)$	**1**	**0.99751**	**0.95346**	**0.81650**	**0.70711**
$T_1(h)$	1	0.99750	0.95000	0.75000	0.50000
$E_1(h)$	1	0.99749	0.94741	0.67564	0.14086
$D_1(h)$	1	0.99751	0.95242	0.80327	0.68394
$S_1(h)$	1	0.99749	0.94509	0.63787	0.11957

Table 2 $g(x) = \frac{1}{\sqrt{1+x}}$, $T_1(h,0,g) = 1 - 1/2h$, $E_1(h,0,g) = 1 - 1/2(e^h - 1)$, $D_1(h,0,g) = 1 - 1/2(1 - e^{-h})$ and $S_1(h,0,g) = \cos(h) - 1/2\sin(h)$.

4.2 Algorithm For Finding The Roots Of Real-Valued Functions

We present a new iterative algorithm for finding the roots of real-valued functions. The derivation of the method is based on the approximation formulae given in the Section 4.1. For example, Newton's method is derived by the approximation

$$f(x^*) \approx f(x_0) + f'(x_0)(x^* - x_0)$$

for a differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $x^* \in \mathbb{R}$ close to $x_0 \in \mathbb{R}$. Let x^* be a root of f and x_0 close to the root. Then, we obtain as an iterative algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{23}$$

for $n \in \mathbb{N}$ and starting point x_0, see for instance [9,10]. We consider the approximation

$$f(x^*) \approx f(x_0) + f'(x_0)(e^{x^*-x_0} - 1)$$

resulting from an exponential infinitesimal fundamentum for $x_0 \in \mathbb{R}$ close to $x^* \in \mathbb{R}$ where x^* is a root of f. Then, we obtain the iteration

$$x_{n+1} = x_n + \ln\left(\left|1 - \frac{f(x_n)}{f'(x_n)}\right|\right) \tag{24}$$

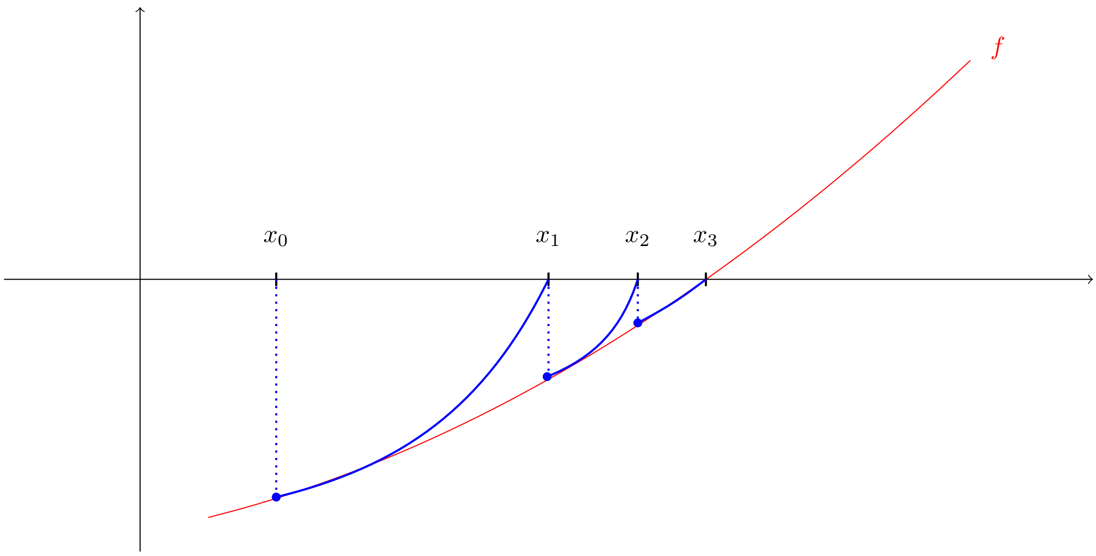

Fig. 12 New method for finding the roots of real-valued functions via an exponential infinitesimal fundamentum.

for $n \in \mathbb{N}$ and starting point x_0, see Figure 12 for an illustration. Note that the convergence of the algorithm as for Newton's method is not ensured for arbitrary functions or arbitrary starting points.

Example 8. We search the square root of two. Therefore, we consider the function $f(x) = x^2 - 2$ and compare Newton's algorithm with the algorithm (24) obtained by an exponential infinitesimal fundamentum in Table 3.

$\sqrt{2} \approx \mathbf{1.4142135624}$	$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + \ln\left(\left\|1 - \frac{f(x_n)}{f'(x_n)}\right\|\right)$
x_0	1	1
x_1	1.5000000000	1.4054651081
x_2	1.4166666667	1.4142025077
x_3	1.4142156863	**1.4142135624**
x_4	**1.4142135624**	**1.4142135624**

Table 3 Comparison of the algorithms for the function $f(x) = x^2 - 2$ and starting point $x_0 = 1$. The new method introduced in (24) converges faster to the root of f than Newton's method (23).

Remark 3. Note that one can obtain in the same way the following iterations via different infinitesimal fundamenta. Using a slow exponential infinitesimal fundamentum (see (19)) we obtain

$$x_{n+1} = x_n - \ln\left(1 + \frac{f(x_n)}{f'(x_n)}\right)$$

for $n \in \mathbb{N}$. Using a sine infinitesimal fundamentum (see (8)) yields the iteration

$$x_{n+1} = x_n + \arctan\left(-\frac{f(x_n)}{f'(x_n)}\right)$$

where arctan is the inverse of the tangent function.

4.3 Numerical Schemes

Approximation of partial differential equations is an important step in proving the existence of solutions. There are several methods of approximation where we mention the use of the convolution with smooth mollifiers, e.g. see [21]. The goal here is to obtain an approximative system by a discretization of partial differential equations. We consider for $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^d$ with $d \in \mathbb{N}$ the Cauchy problem

$$\begin{cases} \partial_t u(x,t) & = \mathbf{G}(u(x,t), \partial_x u(x,t), \partial_x^2 u(x,t), \ldots, \partial_x^m u(x,t)), \\ u(x,0) & = u_0(x) \end{cases} \tag{25}$$

where $m \in \mathbb{N}$, $u_0 : \mathbb{R} \to \mathbb{R}^d$ and $\mathbf{G} : (\mathbb{R}^d)^{m+1} \to \mathbb{R}^d$ are given. Using the method of lines we discretize (25) in time and space by choosing a time step $\Delta t > 0$ and increment $\Delta x > 0$. We define the discrete mesh points by

$$t_n := n\Delta t \quad \text{for} \quad n \in \mathbb{N}_0 \quad \text{and} \quad x_j := j\Delta t \quad \text{for} \quad j \in \mathbb{Z}\,.$$

We approximate the exact solution u of (25) on the discrete mesh points, i.e.

$$u_j^n \approx u(x_j, t_n)\,.$$

Relying on the finite difference method the simplest discretization of (25) is obtained by using a linear infinitesimal fundamentum

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \mathbf{G}\left(u_j^n, \frac{u_{j+1}^n - u_j^n}{\Delta x}, \ldots, \frac{\sum_{k=0}^{m} (-1)^k \binom{m}{k} u_{j+m-k}^n}{(\Delta x)^m} \right) \tag{26}$$

with $u_j^0 = u_0(x_j)$ for $j \in \mathbb{Z}$. Note that there are many different possible choices to discretize (25), e.g. Euler's forward or backward method, Lax Friedrichs and so on, see [20] for an overview about discretization of partial differential equations. The approximative solution is given by solving the discrete system in each time step

$$u_j^{n+1} = u_j^n + \Delta t \mathbf{G}\left(u_j^n, \frac{u_{j+1}^n - u_j^n}{\Delta x}, \ldots, \frac{\sum_{k=0}^{m} (-1)^k \binom{m}{k} u_{j+m-k}^n}{(\Delta x)^m} \right)$$

where $u_j^0 = u_0(x_j)$ is the starting point. We present a new idea for the time discretization of (25) which can be adapted in the space variable. We introduce

$$\frac{u_j^{n+1} - u_j^n}{e^{\Delta t} - 1} = \mathbf{G}\left(u_j^n, \frac{u_{j+1}^n - u_j^n}{\Delta x}, \ldots, \frac{\sum_{k=0}^{m} (-1)^k \binom{m}{k} u_{j+m-k}^n}{(\Delta x)^m} \right)$$

by using an exponential infinitesimal fundamentum. This yields a modified discretization

$$u_j^{n+1} = u_j^n + (e^{\Delta t} - 1)\mathbf{G}\left(u_j^n, \frac{u_{j+1}^n - u_j^n}{\Delta x}, \ldots, \frac{\sum_{k=0}^{m} (-1)^k \binom{m}{k} u_{j+m-k}^n}{(\Delta x)^m} \right). \tag{27}$$

This kind of discretization (27) was used for some specific examples in [22] and is known as the nonstandard finite difference scheme. Note that in this work the advantages or disadvantages of the presented discretizations are not discussed. It is also possible to discretize the equation in the space variable via an exponential infinitesimal fundamentum which yields

$$u_j^{n+1} = u_j^n + (e^{\Delta t} - 1)\mathbf{G}\left(u_j^n, \frac{u_{j+1}^n - u_j^n}{e^{\Delta x} - 1}, \ldots, \frac{\sum_{k=0}^{m} (-1)^k \binom{m}{k} u_{j+m-k}^n}{\sum_{k=0}^{m} \binom{m}{k} (-1)^k e^{k\Delta x}} \right).$$

Note that we can apply in the same way a mixture of infinitesimal fundamenta to obtain discretized version of (25). The choice of the discretization depends on the infinitesimal fundamentum and on the wished property of the discretized system concerning the exactness or discrete energy consistency.

4.4 Entropy Consistent Approximations of Conservation Laws in Time and Space

We present a new approach for discretization of nonlinear conservation laws. This new method yields an explicit numerical scheme which approximates differential equations in time and one-dimensional space in a fully entropy stable way. Entropy stable discretizations in space of time-dependent conservation laws are well known. We continue the discretization of such semi-discretized conservative problems in time and obtain a fully entropy consistent numerical scheme. The method is based on the infinitesimal fundamentum. The infinitesimal fundamentum generalizes the understanding of the derivative for real-valued functions such that it is possible to approximate the derivative in several ways. The suitable choice of the infinitesimal fundamentum depending on the entropy of the system enables the fully entropy consistency of the discretized system.

We introduce the definition of an approximation for partial differential equations. We consider for $\boldsymbol{u} : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^d : (x,t) \mapsto \boldsymbol{u}(x,t)$ with $d \geq 1$ the general system of partial differential equations

$$\begin{aligned} \partial_t \boldsymbol{u}(x,t) &= \mathbf{F}(\boldsymbol{u}(x,t), \partial_x \boldsymbol{u}(x,t), ..., \partial_x^m \boldsymbol{u}(x,t)), \\ \boldsymbol{u}(x,0) &= \boldsymbol{u}_0(x), \end{aligned} \tag{28}$$

where $\mathbf{F} : \mathbb{R}^{d+m+1} \to \mathbb{R}^d$, $m \in \mathbb{N}_0$ and $\boldsymbol{u}_0 : \mathbb{R} \to \mathbb{R}^d$ are given. The first step is to do a spatial discretization of (28) by using the method of lines. We approximate the exact solution $\boldsymbol{u}(x,t)$ of (28) on a spatial grid through

$$\boldsymbol{u}_j(t) \approx \boldsymbol{u}(x_j, t)$$

for $x_j = j\Delta x$ with $j \in \mathbb{Z}$ and a given spatial increment $\Delta x > 0$.

Definition 9. A system for $\boldsymbol{u}_j : \mathbb{R}^+ \to \mathbb{R}^d : t \mapsto \boldsymbol{u}_j(t)$ with $j \in \mathbb{Z}$ of the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{u}_j(t) = \mathbf{F}^*(\Delta x, \boldsymbol{u}_j(t), \boldsymbol{u}_{j-1}(t), \boldsymbol{u}_{j+1}(t), ...), \tag{29}$$

for a given $\mathbf{F}^* : \mathbb{R} \times \left(\mathbb{R}^d\right)^{\mathbb{Z}} \to \mathbb{R}^d$ is called a **semi-discretization** of (28) if the limit case of (29) for $\Delta x \to 0$ coincides with (28), i.e. for a smooth classical solution $\boldsymbol{u}(x,t)$ of (28) it yields that

$$\lim_{h \to 0} \mathbf{F}^*(h, \boldsymbol{u}(x,t), \boldsymbol{u}(x-h,t), \boldsymbol{u}(x+h,t), ...) = \mathbf{F}(\boldsymbol{u}(x,t), \partial_x \boldsymbol{u}(x,t), ..., \partial_x^m \boldsymbol{u}(x,t))$$

for all fixed $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Further a semi-discretization of the form (29) is **conservative** if there are some $\mathbf{F}_{j+1/2} = \mathbf{F}_{j+1/2}(\boldsymbol{u}_j(t), \boldsymbol{u}_{j-1}(t), \boldsymbol{u}_{j+1}(t), ...)$ such that

$$\mathbf{F}^*(\Delta x, \boldsymbol{u}_j(t), \boldsymbol{u}_{j-1}(t), \boldsymbol{u}_{j+1}(t), ...) = \frac{\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}}{\Delta x}$$

for all $j \in \mathbb{Z}$.

Example 9 (Tadmor [27]). We consider for the scalar equation

$$u_t(x,t) = u_x(x,t) \exp(u(x,t)) =: \mathbf{F}(u(x,t), u_x(x,t)) \tag{30}$$

the system of ordinary differential equations

$$\dot{u}_j(t) = \frac{\exp(u_{j+1}(t)) - \exp(u_{j-1}(t))}{2\Delta x} =: \mathbf{F}^*(\Delta x, u_{j-1}(t), u_{j+1}(t)), \tag{31}$$

where we replaced the time derivative of $u_j(t)$ by $\dot{u}_j(t)$. Then it yields for a smooth solution $u(x,t)$ of (30) by using the rule of L'Hospital that

$$\begin{aligned}\lim_{h\to 0}\mathbf{F}^*(h, u(x-h,t), u(x+h,t)) &= \lim_{h\to 0}\frac{u_x(x+h,t)\exp(u_x(x+h,t))}{2}\\ &\quad+\lim_{h\to 0}\frac{u_x(x-h,t)\exp(u_x(x-h,t))}{2}\\ &= \mathbf{F}(u(x,t), u_x(x,t))\end{aligned}$$

which shows that (31) is a semi-discretization of (30). Further (31) is in conservative form by choosing

$$\mathbf{F}_{j+1/2}(u_j(t), u_{j-1}(t), u_{j+1}(t), ...) = \frac{\exp(u_{j+1}(t)) + \exp(u_j(t))}{2}.$$

The second step is to go on with the discretization in time. We approximate the time derivative in (29) to obtain a fully discretized system. That means we approximate the exact solution $\boldsymbol{u}(x,t)$ of (28) through

$$\boldsymbol{u}_j^n \approx \boldsymbol{u}(x_j, t_n)$$

for $t_n = n\Delta t$ with $n \in \mathbb{N}_0$ and a given time increment $\Delta t > 0$.

Definition 10. A system for $\boldsymbol{u}_j^n \in \mathbb{R}^d$ with $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$ of the form

$$\mathbf{G}^*(\Delta t, \boldsymbol{u}_j^{n+1}, \boldsymbol{u}_j^n) = \mathbf{F}^*(\Delta x, \boldsymbol{u}_j^n, \boldsymbol{u}_{j-1}^n, \boldsymbol{u}_{j+1}^n, ...), \tag{32}$$

with a given $\mathbf{G}^* : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is called an **approximation** of (28) if the limit case of (32) for $\Delta t \to 0$ and $\Delta x \to 0$ coincides with (28), i.e. for a smooth classical solution $\boldsymbol{u}(x,t)$ of (28) it yields that

$$\lim_{\sigma\to 0}\mathbf{G}^*(\sigma, \boldsymbol{u}(x, t+\sigma), \boldsymbol{u}(x,t)) = \boldsymbol{u}_t(x,t)$$

and

$$\lim_{h\to 0}\mathbf{F}^*(h, \boldsymbol{u}(x,t), \boldsymbol{u}(x-h,t), \boldsymbol{u}(x+h,t), ...) = \mathbf{F}(\boldsymbol{u}(x,t), \partial_x\boldsymbol{u}(x,t), ..., \partial_x^m\boldsymbol{u}(x,t))$$

for all fixed $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. Further an approximation of (28) is called **exlicit** if (32) can be written in the form

$$\boldsymbol{u}_j^{n+1} = \mathbf{G}(\Delta t, \Delta x, \boldsymbol{u}_j^n, \boldsymbol{u}_{j-1}^n, \boldsymbol{u}_{j+1}^n, ...)$$

for all $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$ for some $\mathbf{G} : \mathbb{R}^+ \times \mathbb{R}^+ \times (\mathbb{R}^d)^{\mathbb{Z}} \to \mathbb{R}^d$.

Example 10. We consider for (30) an approximation of the form

$$\exp(-u_j^n)\frac{\exp(u_j^{n+1}) - \exp(u_j^n)}{\Delta t} = \frac{\exp(u_{j+1}^n(t)) - \exp(u_{j-1}^n(t))}{2\Delta x} \tag{33}$$

for all $n \in \mathbb{N}_0$ and $j \in \mathbb{Z}$. Then it yields that (33) is an approximation of (30) since we obtain for a smooth solution $u(x,t)$ of (30) by using the rule of L'Hospital that

$$\begin{aligned}\lim_{\sigma\to 0}\mathbf{G}^*(\sigma, u(x,t+\sigma), u(x,t)) &= \lim_{\sigma\to 0}\exp(-u(x,t))\frac{\exp(u(x,t+\sigma)) - \exp(u(x,t))}{\sigma}\\ &= \lim_{\sigma\to 0}\frac{u_t(x,t+\sigma)\exp(u(x,t+\sigma))}{\exp(u(x,t))} = u_t(x,t)\end{aligned}$$

and the limit $\lim_{h\to 0}\mathbf{F}^*(h, u(x,t), u(x-h,t), u(x+h,t), ...)$ was already calculated in Example 9. Further the approximation (33) is explicit since (33) can be written in the form

$$u_j^{n+1} = u_j^n + \ln\left(1 + \frac{\Delta t}{2\Delta x}\Big(\exp(u_{j+1}^n) - \exp(u_{j-1}^n)\Big)\right).$$

Note that the time approximation on the left in (33) is done by using the funtion $f(u) = \exp(u)$ which can be generalized to

$$\frac{1}{f'(u_j^n)} \frac{f(u_j^{n+1}) - f(u_j^n)}{\Delta t}.$$

Now if we choose some entropy function η corresponding to (28) for the time approximation, we obtain an energy/entropy consistent numerical scheme. For example, consider the scalar equation

$$u_t(x,t) - u(x,t)u_x(x,t) = 0 \tag{34}$$

and the entropy function $\eta_1(u) = \ln(u)$ to (34). By using the Tadmor [27] approximation in space, we consider the space approximation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_j(t) = u_j(t) \frac{u_{j+1}(t) - u_{j-1}(t)}{2\Delta x}. \tag{35}$$

Using the entropy approximation in time we obtain an energy consistent approximation of (34) as

$$u_j^{n+1} = u_j^n \exp\left(\frac{\Delta t}{2\Delta x} \left(u_{j+1}^n - u_{j-1}^n \right) \right) \tag{36}$$

which satisfies

$$\eta_1(u_j^{n+1}) - \eta_1(u_j^n) \leq Q_{j+1}^n - Q_{j-1}^n$$

with $Q_j^n = \frac{\Delta t}{2\Delta x} u_j^n$. In the same way we can obtain an entropy consistent approximation for (34) by using the entropy function $\eta_2(u) = \frac{u^2}{2}$. Again, using Tadmor in space but not with the same discretization as before, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} u_j(t) = \frac{u_{j+1}(t) + u_j(t) + u_{j-1}(t)}{3} \cdot \frac{u_{j+1}(t) - u_{j-1}(t)}{2\Delta x} \tag{37}$$

as a space approximation of (34). We obtain by using η_2 in the time discretization

$$u_j^{n+1} = \sqrt{(u_j^n)^2 + \frac{\Delta t}{3\Delta x} u_j^n (u_{j+1}^n + u_j^n + u_{j-1}^n)(u_{j+1}^n - u_{j-1}^n)} \tag{38}$$

which satisfies $\eta_2(u_j^{n+1}) - \eta_2(u_j^n) \leq W_{j+1}^n - W_j^n$ for some explicit given W_j^n.

We note that this mindset for approximation of partial differential equations can maybe be used for showing the existence of solution to the original problem by using a suitable approximation and entropy function in some Sobolev spaces.

4.5 Series

We consider a concrete example such that the rest term in the expansion formulae (12) and (22) vanishes. This yields a representation of the considered function as an infinite sum. For instance, the exponential function results in (via Taylor's expansion formula (9) with expansion point $x_0 = 0$)

$$e^h = \sum_{j=0}^{\infty} \frac{h^j}{j!}$$

for $h \in \mathbb{R}$. We apply the new kind of expansion formulae (12) and (22) to a simple function to obtain convergent series. We consider the function $f(x) = x$. Then, Theorem 1 (respectively equation (12)) yields the expansion of f for $x \in \mathbb{R}$ and some $h \in \mathbb{R}$

$$f(x+h) = x + h = x + (e^h - 1) - \frac{1}{2}(e^h - 1)^2 + \frac{2}{3!}(e^h - 1)^3 - \frac{6}{4!}(e^h - 1)^4 + \cdots$$

which implies

$$h = (e^h - 1) - \frac{1}{2}(e^h - 1)^2 + \frac{1}{3}(e^h - 1)^3 - \frac{1}{4}(e^h - 1)^4 + \cdots .$$

Choosing $h = \ln(y)$ for some $y > 0$ yields the well-known identity

$$\ln(y) = (y-1) - \frac{1}{2}(y-1)^2 + \frac{1}{3}(y-1)^3 - \frac{1}{4}(y-1)^4 + \cdots = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(y-1)^j$$

where the sum is convergent for all $0 < y \leq 2$. It is also possible to consider the expansion formula given in Theorem 2 (respectively equation (22)) which yields in the same way

$$h = (1-e^{-h}) + \frac{1}{2}(1-e^{-h})^2 + \frac{1}{3}(1-e^{-h})^3 + \frac{1}{4}(1-e^{-h})^4 + \cdots .$$

Choosing again $h = \ln(y)$ implies

$$\ln(y) = \sum_{j=1}^{\infty} \frac{(1-\frac{1}{y})^j}{j}$$

which is convergent for all $y \geq 1/2$. Altogether, we represent the natural logarithm $\ln(y)$ for all $y > 0$ as

$$\ln(y) = \begin{cases} \sum\limits_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(y-1)^j & \text{for} \quad 0 < y \leq 1/2, \\ \sum\limits_{j=1}^{\infty} \frac{1}{j}\left(1-\frac{1}{y}\right)^j & \text{for} \quad 1/2 \leq y, \end{cases} \tag{39}$$

where we note that the identity

$$\ln(y) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j}(y-1)^j = \sum_{j=1}^{\infty} \frac{1}{j}\left(1-\frac{1}{y}\right)^j$$

holds for values $1/2 \leq y \leq 2$. The representation of the natural logarithm $\ln(y)$ for $y > 1$ can also be obtained by the identity

$$\ln(1+y) = -\ln\left(1 - \frac{y}{1+y}\right) .$$

The inequality $|-\frac{y}{1+y}| < 1$ implies

$$\ln(1+y) = -\ln\left(1-\frac{y}{1+y}\right) = \sum_{j=1}^{\infty} \frac{(-1)^j}{j}\left(-\frac{y}{1+y}\right)^j = \sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{y}{1+y}\right)^j$$

such that

$$\ln(z) = \sum_{j=1}^{\infty} \frac{1}{j}\left(1-\frac{1}{z}\right)^j$$

for all $z \geq 2$ which verifies the representation (39).

We mention that by using several expansion formulae ((9), (12) and (22)) we can solve ordinary and partial differential equations. In particular, the representation of functions as series (e.g. (39)) allows to solve differential equations. We illustrate this approach on the following example. We consider the one-dimensional problem for $t \in [0, 1/e)$

$$x'(t) = e^{x(t)}, \quad x(0) = 1. \tag{40}$$

By using the separation of variables the solution to (40) on the interval $[0, 1/e)$ is given by $x(t) = 1 - \ln(1-et)$. We state another possible way to solve the problem (40). We expand the solution $x(t)$ to (40) via Taylor's formula (9) for the expansion point $x_0 = 0$. Differentiating the first equation in (40) with respect to t yields the values of the derivatives of the solution $x(t)$ at the expansion point. We obtain

$$x(h) = 1 + eh + \frac{1}{2}e^2h^2 + \cdots = 1 + \sum_{j=1}^{\infty} \frac{(eh)^j}{j} . \tag{41}$$

By applying (39) to the sum in (41) we obtain $x(h) = 1 + \ln\left(\frac{1}{1-eh}\right)$.

5 Open Questions And Conjectures

We list open questions and some conjectures:

1. We think that Weierstrass approximation theorem (see e.g. [30, 5, 29]) can be adapted such that every continuous functions can be uniformly approximated as closely as desired by a function of the form
$$\sum_{j=0}^{n} c_{j,n}(e^x - 1)^j$$
for $n \in \mathbb{N}_0$ and $c_{j,n} \in \mathbb{R}$. The proof of the following statement is pending.

   Conjecture (Approximation of continuous functions). *Let $f \in C([a,b])$ for $a, b \in \mathbb{R}$ with $a < b$. Then*
$$\forall \epsilon > 0 \, \exists n \in \mathbb{N}_0 \, \exists E_n \, \forall x \in [a,b] : |f(x) - E_n(x)| < \epsilon \, ,$$
where $E_n(x)$ is for $n \in \mathbb{N}_0$ a function of the form
$$E_n(x) = \sum_{j=0}^{n} c_{j,n}(e^x - 1)^j$$
for $c_{j,n} \in \mathbb{R}$.
2. In Theorem 1 (see (11)) we present a recursive formula for the coefficients a_k^n. It is expected to have an explicit representation for a_k^n. That is, there is some
$$\Lambda : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$$
such that $\Lambda(n,k) = a_k^n$ for all $(n,k) \in \mathbb{N}^2$. This property would simplify the series representation in the expansion formula in Theorem 1.
3. A generalized version for the expansion formula is missing. That means, we expect an expansion formula depending on an arbitrary infinitesimal fundamentum g. The first step in this matter is to consider the generalized exponential infinitesimal fundamentum in $x_0 \in \mathbb{R}$ with the infinitesimal dh of the form
$$g^*(z) = \frac{f^*(x_0 + dh) - f(x_0)}{e^{\lambda dh} - 1} \left(e^{\lambda(z - x_0)} - 1\right) + f(x_0)$$
for $\lambda \in \mathbb{R} \setminus \{0\}$. This should yield a generalized version of (12) respectively (22) depending on λ. Then, the expansion formulae in Theorem 1 (respectively equation (12)) and Theorem 2 (respectively equation (22)) are obtained by the special cases for $\lambda = 1$ and $\lambda = -1$.
4. The approximation formula resulting by a sine infinitesimal fundamentum
$$f(x_0 + h) \approx \cos(h) f(x_0) + f'(x_0) \sin(h)$$
indicates also an expansion formula. This formula is pending.
5. This works deals with the one-dimensional case. As Taylor's Theorem, the expansion formulae derived in this work and also the concept of the infinitesimal fundamentum should be extended into the multi-dimensional case.
6. Which conditions on the smooth real-valued function f have to be put for the convergence of the expansion formula in Theorem 1 or Theorem 2? The convergence of the resulting series for explicit functions should be considered in a general framework.
7. Is it possible to define fractional derivatives (see e.g. [24]) via modified infinitesimal fundamenta?
8. Albert Einstein's theory of special relativity deduces the kinetic energy of a moving object with velocity v and mass m as
$$\mathrm{E}_{\mathrm{kin}} = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} - mc^2$$
where c is the speed of light, see [16]. Using Taylor's Theorem for $v \ll c$ yields in approximation for the kinetic energy
$$\mathrm{E}_{\mathrm{kin}} \approx \frac{1}{2} mv^2$$
as in classical Newtonian physics where we used $(1+x)^{-1/2} = 1 - 1/2x + 3/8x^2 - \cdots$ for $|x| \ll 1$. Is this approach applicable for other physical examples by using the expansion formula from Theorem 1 or Theorem 2 or the approximation formulae given in Section 4.1?

A Proof Of Theorem 2

Let $a,b \in \mathbb{R}$ with $a<b$ and $n \in \mathbb{N}_0$. Further, let $f \in C^n([a,b])$ and $f^{(n)}$ differentiable in (a,b). Then, there is a constant $c \in (a,b)$ such that

$$f(b) = f(a) + \sum_{j=1}^{n} \frac{(1-e^{a-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(a) + R_n(h,a,c)\,,$$

where

$$R_n(a-b,a,c) = e^{b-c}\frac{(1-e^{a-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(c)$$

and $j,k \in \mathbb{N}$ with $k \le j$

$$b_k^j := \begin{cases} 1 & \text{for} \quad k=1, j \ge 1\,, \\ (j-1)b_{k-1}^{j-1} + b_k^{j-1} & \text{for} \quad 1<k<j\,, \\ (j-1)! & \text{for} \quad j=k \ge 1\,. \end{cases}$$

Proof. By following the proof of Theorem 1 we define

$$F(t) := f(b) - f(t) - \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) - m\frac{(1-e^{t-b})^{n+1}}{(n+1)!}$$

with

$$m := \frac{f(b) - f(a) - \sum_{j=1}^{n} \frac{(1-e^{a-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(a)}{\frac{(1-e^{a-b})^{n+1}}{(n+1)!}}\,.$$

Then, we obtain

$$F(b) = 0 = F(b)$$

and the mean value theorem yields some $c \in (a,b)$ such that $F'(c)=0$. We determine the derivative of F

$$\begin{aligned} F'(t) \,=\, & -f'(t) - \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) \\ & + e^{t-b} \sum_{j=1}^{n} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) + me^{t-b}\frac{(1-e^{t-b})^n}{n!}\,. \end{aligned}$$

For all $n \in \mathbb{N}$ we have the relation

$$\begin{aligned} -f'(t) &- \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\ &= -\frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t) \end{aligned} \tag{42}$$

which will be proved as before by an induction.
Induction Basis $n=1$: We obtain

$$\begin{aligned} -f'(t) - (1-e^{t-b})b_1^1 f''(t) + e^{t-b} b_1^1 f'(t) &= -(1-e^{t-b})(f''(t) + f'(t)) \\ &= -(1-e^{t-b})(b_1^2 f''(t) + b_2^2 f'(t)) \end{aligned}$$

which shows that (42) is true for $n=1$.
Induction Hypothesis: We assume that statement (42) holds for $n \in \mathbb{N}$, i.e.

$$\begin{aligned} -f'(t) &- \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\ &= -\frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t)\,. \end{aligned} \tag{43}$$

By using the induction hypothesis (43) we show that the statement (42) holds for $n+1$.
Inductive Step $n \rightsquigarrow n+1$**:** We calculate

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -f'(t) - \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&\quad - \frac{(1-e^{t-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+3-k)}(t) + e^{t-b} \frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t) \,.
\end{aligned}$$

Using the induction hypothesis (43) yields

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t) - \frac{(1-e^{t-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+3-k)}(t) \\
&\quad + e^{t-b} \frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t) \,.
\end{aligned} \tag{44}$$

We take on the right hand-side in (44) the first and last term together. Further, we factor out $-\frac{(1-e^{t-b})^{n+1}}{(n+1)!}$ which implies

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(\sum_{k=1}^{n+1} b_k^{n+1} f^{(n+3-k)}(t) + (n+1) b_k^{n+1} f^{(n+2-k)}(t) \right) .
\end{aligned}$$

A shifting in the index yields

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(\sum_{k=0}^{n} b_{k+1}^{n+1} f^{(n+2-k)}(t) + \sum_{k=1}^{n+1} (n+1) b_k^{n+1} f^{(n+2-k)}(t) \right) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(\sum_{k=1}^{n} b_{k+1}^{n+1} f^{(n+2-k)}(t) + \sum_{k=1}^{n} (n+1) b_k^{n+1} f^{(n+2-k)}(t) \right) \\
&\quad - \frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(b_1^{n+1} f^{(n+2)}(t) + (n+1) b_{n+1}^{n+1} f'(t) \right) .
\end{aligned}$$

We obtain

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(b_1^{n+1} f^{(n+2)}(t) + (n+1) b_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n} f^{(n+2-k)}(t) \left(b_{k+1}^{n+1} + (n+1) b_k^{n+1} \right) \right) .
\end{aligned}$$

Using

$$\left(b_{k+1}^{n+1} + (n+1) b_k^{n+1} \right) = b_{k+1}^{n+2}$$

and shifting again in the index yields

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(b_1^{n+1} f^{(n+2)}(t) + (n+1) b_{n+1}^{n+1} f'(t) + \sum_{k=2}^{n+1} b_k^{n+2} f^{(n+3-k)}(t) \right) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left(b_1^{n+1} f^{(n+2)}(t) + (n+1) b_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n+1} b_k^{n+2} f^{(n+3-k)}(t) - b_1^{n+2} f^{(n+2)}(t) \right) .
\end{aligned}$$

Using $b_1^{n+1} = b_1^{n+2}$ and changing the limit in the sum yields

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \left((n+1) b_{n+1}^{n+1} f'(t) + \sum_{k=1}^{n+2} b_k^{n+2} f^{(n+3-k)}(t) - b_{n+2}^{n+2} f'(t) \right) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} b_k^{n+2} f^{(n+3-k)}(t) + \frac{(1-e^{t-b})^{n+1}}{(n+1)!} f'(t) \left(-(n+1) b_{n+1}^{n+1} + b_{n+2}^{n+2} \right) .
\end{aligned}$$

Using $b_{n+1}^{n+1} = \ n!$ and $b_{n+2}^{n+2} = (n+1)!$ yields

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} b_k^{n+2} f^{(n+3-k)}(t) + \frac{(1-e^{t-b})^{n+1}}{(n+1)!} f'(t) \big((n+1)! - (n+1)! \big) .
\end{aligned}$$

End of Induction: Altogether, we obtain

$$\begin{aligned}
&-f'(t) - \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) + e^{t-b} \sum_{j=1}^{n+1} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) \\
&= -\frac{(1-e^{t-b})^{n+1}}{(n+1)!} \sum_{k=1}^{n+2} b_k^{n+2} f^{(n+3-k)}(t)
\end{aligned}$$

which completes the induction and we obtain (42) for all $n \in \mathbb{N}$. That means for the derivative of F

$$\begin{aligned}
F'(t) &= -f'(t) - \sum_{j=1}^{n} \frac{(1-e^{t-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+2-k)}(t) \\
&\quad + e^{t-b} \sum_{j=1}^{n} \frac{(1-e^{t-b})^{j-1}}{(j-1)!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(t) + m e^{t-b} \frac{(1-e^{t-b})^n}{n!} \\
&= -\frac{(1-e^{t-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(t) + m e^{t-b} \frac{(1-e^{t-b})^n}{n!} .
\end{aligned}$$

The mean value theorem yields some $c \in (a, b)$ with $F'(c) = 0$ which means

$$0 = F'(c) = -\frac{(1-e^{c-b})^n}{n!} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(c) + m e^{c-b} \frac{(1-e^{c-b})^n}{n!} . \tag{45}$$

Equation (45) implies

$$m = e^{b-c} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(c) .$$

By using $F(a) = 0$ we obtain

$$f(b) = f(a) + \sum_{j=1}^{n} \frac{(1-e^{a-b})^j}{j!} \sum_{k=1}^{j} b_k^j f^{(j+1-k)}(a) + e^{b-c} \sum_{k=1}^{n+1} b_k^{n+1} f^{(n+2-k)}(c) \frac{(1-e^{a-b})^{n+1}}{(n+1)!}$$

which completes the proof of Theorem 2. □

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